CHAPTER 1
SOME PRELIMINARIES

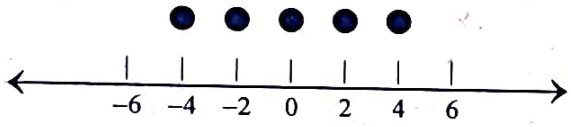
Real number line, Solution of equations involving absolute values, Inequalities.

Exercises 1.1

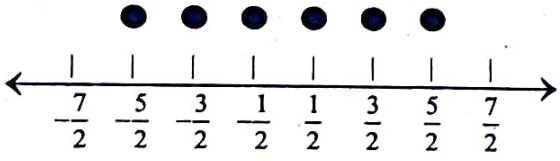
In Exercises (1-10), graph the following real numbers on some real line by choosing suitable scale.

Solutions:

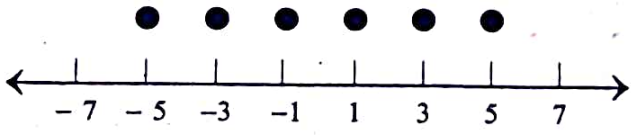
1. $-4,-2,0,2,4$



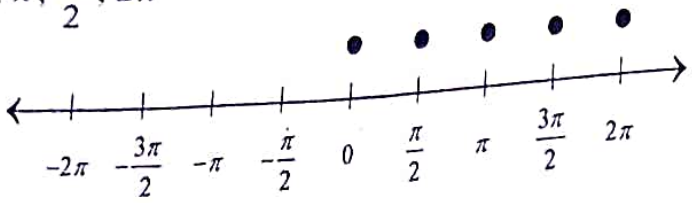
2. $-\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2}$



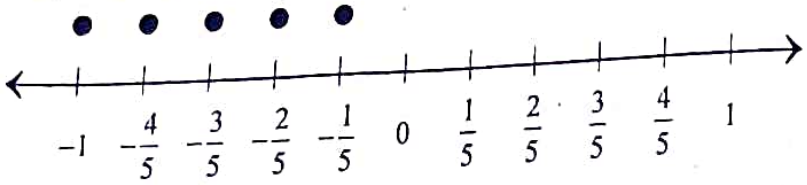
3. $\pm 1, \pm 3, \pm 5$



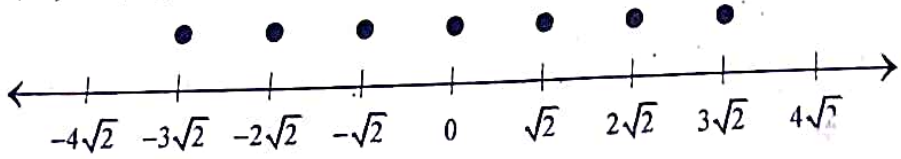
4. $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$



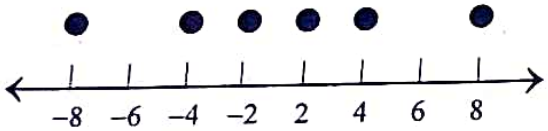
5. $-\frac{1}{5}, -\frac{2}{5}, -\frac{3}{5}, -\frac{4}{5}, -1$



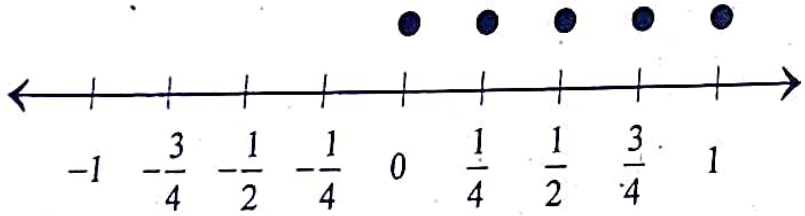
6. $-3\sqrt{2}, -2\sqrt{2}, -\sqrt{2}, 0, \sqrt{2}, 2\sqrt{2}, 3\sqrt{2}$



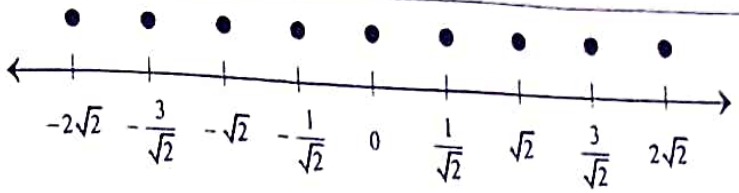
7. $\pm 2, \pm 4, \pm 8$



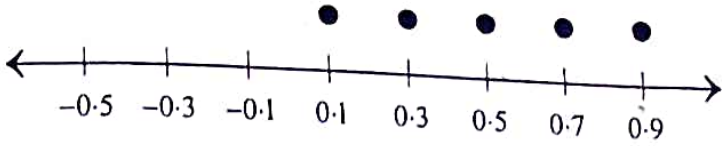
8. $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$



9. $0, \pm\frac{1}{\sqrt{2}}, \pm\sqrt{2}, \pm\frac{3}{\sqrt{2}}, \pm 2\sqrt{2}$

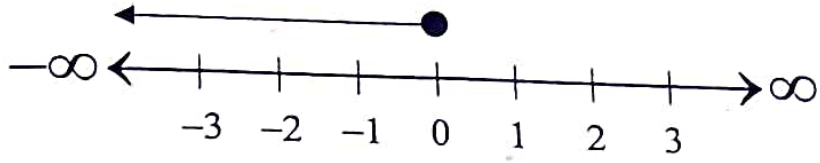


10. $0{\cdot}1, 0{\cdot}3, 0{\cdot}5, 0{\cdot}7, 0{\cdot}9$

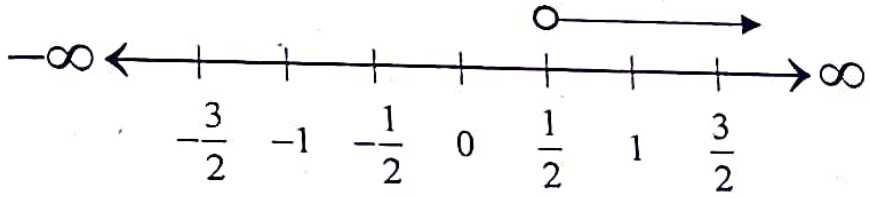


In Exercises (11-16), sketch the following rays. Also identify an open or closed ray.

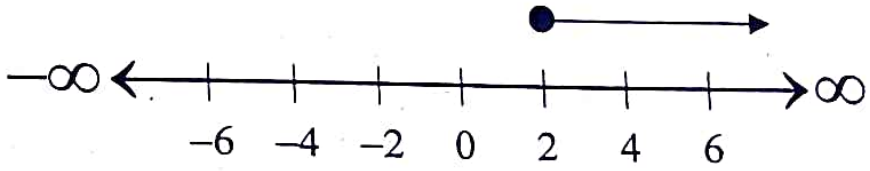
11. $(-\infty, 0]$



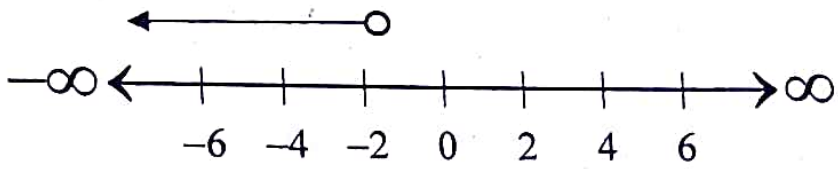
12. $\left(\frac{1}{2}, \infty\right)$



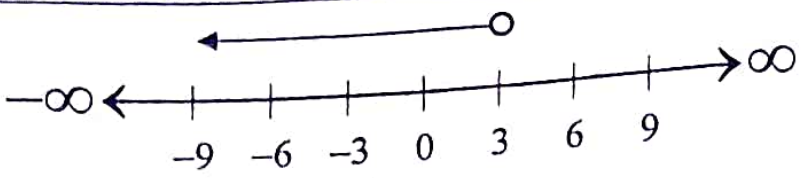
13. $[2, \infty)$



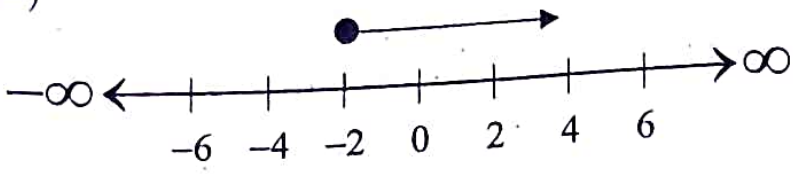
14. $(-\infty, -2)$



15. $(-\infty, 3)$

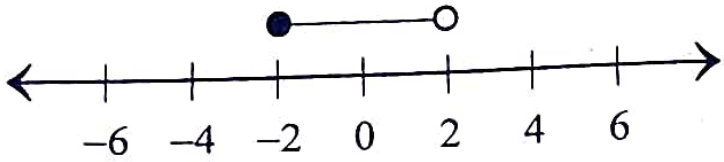


16. $[-2,\infty)$

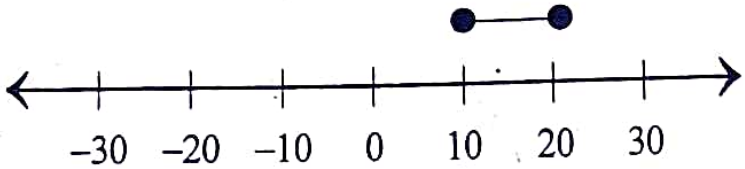


Sketch the following intervals $(17-24)$.

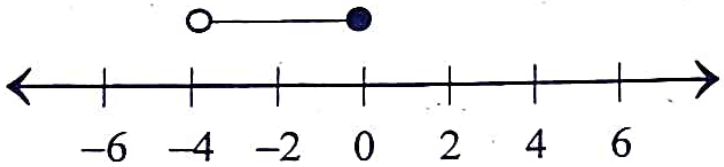
17. $[-2,2)$



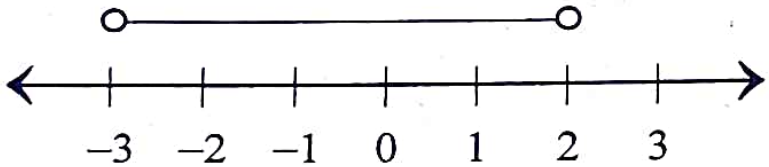
18. $[10,20]$



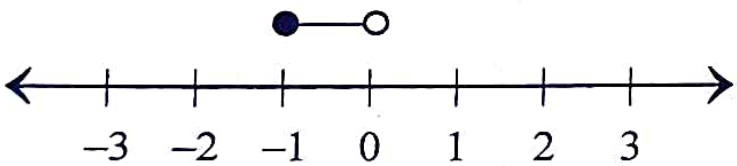
19. $(-4,0]$



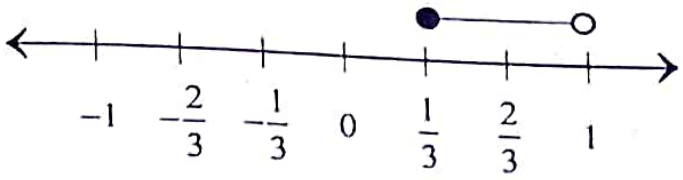
20. $(-3,2)$



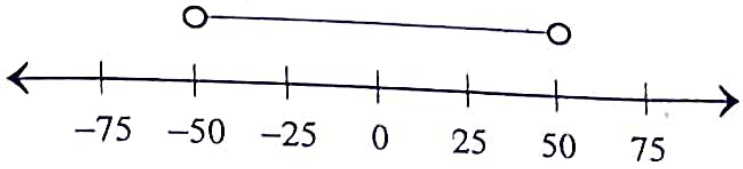
21. $[-1,0)$



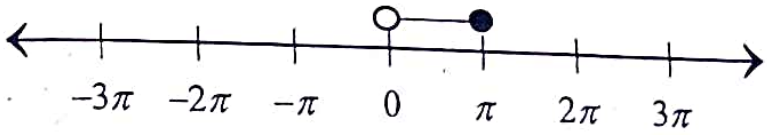
22. $\left[\frac{1}{3}, 1\right)$



23. $(-50, 50)$



24. $(0, \pi]$



Exercises 1.2

Solve the following equations.

Solutions:

1. $|x| = 6$

   $x = \pm 6$

   S.set $= \{6, -6\}$

2. $|x| = -4$

   No solution because absolute value of any real number is always non-negative.

   S. set $= \{\ \}$.

3. $|x-6| = 3$ (1)

   $x - 6 = \pm 3$

   $x - 6 = 3$ or $x - 6 = -3$

   $x = 9$ or $x = 3$

   Checking

   Put $x = 9$ in (1): $|9-6| = 3$, $|3| = 3$, $3 = 3$, True

   Put $x = 3$ in (1): $|3-6| = 3$, $|-3| = 3$, $3 = 3$, True

   S. set $= \{9, 3\}$.

4. $|x-2| = 4$ (1)

   $x - 2 = \pm 4$

   $x - 2 = 4$ or $x - 2 = -4$

   $x = 6$ or $x = -2$

   Checking

   Put $x = 6$ in (1): $|6-2| = 4$, $|4| = 4$, $4 = 4$, True

   Put $x = -2$ in (1): $|-2-2| = 4$, $|-4| = 4$, $4 = 4$, True

   S. set $= \{6, -2\}$

5. $|x+3| = 7$ (1)

   $x + 3 = \pm 7$

   $x + 3 = 7$ or $x + 3 = -7$

$x=4$ or $x=-11$

Checking

Put $x=4$ in (1) Put $x=-11$ in (1).

$|4+3|=7$ $|-11+3|=7$

$|7|=7$ $|-7|=7$

$7=7$, True $7=7$, True

S. set $=\{4,-11\}$.

6. $|2x-7|=1$ (1)

$2x-7=\pm 1$

$2x-7=1$ or $2x-7=-1$

$2x=8$ or $2x=6$

$x=4$ or $x=3$

Checking

Put $x=4$ in (1) Put $x=3$ in (1)

$|2(4)-7|=1$ $|2(3)-7|=1$

$|1|=1$ $|-1|=1$

$1=1$, True $1=1$, True

S. set $=\{4,3\}$.

7. $|2x-6|=4$ (1)

$2x-6=\pm 4$

$2x-6=4$ or $2x-6=-4$

$2x=10$ or $2x=2$

$x=5$ or $x=1$

Checking

Put $x=5$ in (1) Put $x=1$ in (1)

$|2(5)-6|=4$ $|2(1)-6|=4$

$|4|=4$ $|-4|=4$

$4=4$, True $4=4$, True

S. set $=\{5,1\}$.

8. $\frac{1}{2}|3x+10|=5$ (1)

$|3x+10|=10$

$3x+10=\pm 10$

$3x+10=10$ or $3x+10=-10$

$3x=0$ or $3x=-20$

$x=0$ or $x=\frac{-20}{3}$

Checking

Put $x=0$ in (1), Put $x=\frac{-20}{3}$ in (1)

$\frac{1}{2}|3(0)+10|=5$, $\frac{1}{2}\left|3\left(\frac{-20}{3}\right)+10\right|=5$

$\frac{1}{2}|10|=5$ $\frac{1}{2}|-10|=5$

$5=5$, True $5=5$, True

S. set $=\left\{0,\frac{-20}{3}\right\}$.

9. $|10-2x|=|x+5|$ (1)

$10-2x=\pm(x+5)$

$10-2x=x+5$ or $10-2x=-(x+5)$

$-3x=-5$ or $10-2x=-x-5$

$x=\frac{5}{3}$, True or $-x=-15$

$x=15$, True

Checking

Put $x=\frac{5}{3}$ in (1) Put $x=15$ in (1)

$\left|10-2\left(\frac{5}{3}\right)\right|=\left|\frac{5}{3}+5\right|$, $|10-2(15)|=|15+5|$

$\left|10-\frac{10}{3}\right|=\left|\frac{5}{3}+5\right|$, $|10-30|=|20|$

$\left|\frac{20}{3}\right|=\left|\frac{20}{3}\right|$, $|-20|=|20|$

$\frac{20}{3}=\frac{20}{3}$, True $20=20$, True

S. set $=\left\{\frac{5}{3},15\right\}$.

10. $|x+3|=|5-x|$ (1)

$x+3=\pm(5-x)$

$x+3=5-x$ or $x+3=-(5-x)$

$2x=2$ or $x+3=-5+x$

$x=1$ or $0=8$, Not admissible

Checking

Put $x=1$ in (1)

$|1+3|=|5-1|$

$|4|=|4|$

$4=4$, True

S. set $=\{1\}$.

11. $|3x-10|=|2x-7|$ (1)

$3x-10=\pm(2x-7)$

$3x-10=2x-7$ or $3x-10=-(2x-7)$

$x=10-7$ or $3x-10=-2x+7$

$x=3$ or $5x=17$

$x=\frac{17}{5}$

Checking

Put $x=3$ in (1) Put $x=\frac{17}{5}$ in (1)

$|3(3)-10|=|2(3)-7|$, $\left|3\left(\frac{17}{5}\right)-10\right|=\left|2\left(\frac{17}{5}\right)-7\right|$

$|9-10|=|6-7|$ $\left|\frac{51}{5}-10\right|=\left|\frac{34}{5}-7\right|$

$|-1|=|-1|$ $\left|\frac{1}{5}\right|=\left|-\frac{1}{5}\right|$

$1=1$, True $\frac{1}{5}=\frac{1}{5}$, True

S. set $=\left\{3,\frac{17}{5}\right\}$.

12. $|5-3x|=|-2x+7|$ (1)

$5-3x=\pm(-2x+7)$

$5-3x=-2x+7$ or $5-3x=-(-2x+7)$

$-x=2$ or $5-3x=2x-7$

$x=2$ or $-5x=-12$

$x=\frac{12}{5}$

Checking

Put $x=2$ in (1) Put $x=\frac{12}{5}$ in (1)

$|5-3(2)|=|-2(2)+7|$ $\left|5-3\left(\frac{12}{5}\right)\right|=\left|-2\left(\frac{12}{5}\right)+7\right|$

$|-1|=|3|$ $\left|5-\frac{36}{5}\right|=\left|-\frac{24}{5}+7\right|$

$1=3$, False $\left|-\frac{11}{5}\right|=\left|\frac{11}{5}\right|$

$\frac{11}{5}=\frac{11}{5}$, True

Since $x=2$ does not satisfy the original equation, therefore

S. set $=\left\{\frac{12}{5}\right\}$.

13. $|x|=|-x+5|$ (1)

$x=\pm(-x+5)$

$x=-x+5$ or $x=-(-x+5)$

$2x=5$ or $x=x+5$

$x=\frac{5}{2}$ or $0=5$, Not admissible

Checking

Put $x=\frac{5}{2}$ in (1)

$\left|\frac{5}{2}\right|=\left|-\frac{5}{2}+5\right|$

$\frac{5}{2}=\frac{5}{2}$, True

S. set $=\left\{\frac{5}{2}\right\}$.

14. $|x+2|-3=5-|x+2|$ (1)

$|x+2|+|x+2|=5+3$

$2|x+2|=8$

$|x+2|=4$

$x+2=\pm4$

$x+2=4$ or $x+2=-4$

$x=2$ or $x=-6$

Checking

Put $x=2$ in (1) Put $x=-6$ in (1)

$|2+2|-3=5-|2+2|$ $|-6+2|-3=5-|-6+2|$

$4-3=5-4$ $|-4|-3=5-|-4|$

$1=1$, True $4-3=5-4$

$1=1$, True

S. set $=\{2,-6\}$.

15. $\frac{1}{2}|3x+2|-4=11$ (1)

$\frac{1}{2}|3x+2|=15$

$|3x+2|=30$

$3x+2=\pm 30$

$3x+2=30$ or $3x+2=-30$

$3x=28$ or $3x=-32$

$x=\frac{28}{3}$ or $x=-\frac{32}{3}$

Checking

Put $x=\frac{28}{3}$ in (1) Put $x=\frac{-32}{3}$ in (1)

$\frac{1}{2}\left|3\left(\frac{28}{3}\right)+2\right|-4=11$, $\frac{1}{2}\left|3\left(-\frac{32}{3}\right)+2\right|-4=11$

$\frac{1}{2}|28+2|-4=11$ $\frac{1}{2}|-32+2|-4=11$

$\frac{1}{2}(30)-4=11$ $\frac{1}{2}(30)-4=11$

$11=11$, True $11=11$, True

S. set $=\left\{\frac{28}{3},-\frac{32}{3}\right\}$.

16. $\left|\frac{x+3}{4-x}\right|=3$ (1)

$\frac{x+3}{4-x}=\pm 3$

$x+3=\pm 3(4-x)$

$x+3=3(4-x)$ or $x+3=-3(4-x)$

$x+3=12-3x$ or $x+3=-12+3x$

$4x=9$ or $-2x=-15$

$x=\frac{9}{4}$ or $x=\frac{15}{2}$

Checking

Put $x=\frac{9}{4}$ in (1) Put $x=\frac{15}{2}$ in (1)

$\left|\frac{\frac{9}{4}+3}{4-\frac{9}{4}}\right|=3$ $\left|\frac{\frac{15}{2}+3}{4-\frac{15}{2}}\right|=3$

$\left|\frac{9+12}{16-9}\right|=3$ $\left|\frac{15+6}{8-15}\right|=3$

$\left|\frac{21}{7}\right|=3$ $\left|\frac{21}{-7}\right|=3$

$3=3$, True $3=3$, True

S. set $=\left\{\frac{9}{4},\frac{15}{2}\right\}$.

Exercises 1.3

Solve the following inequalities $(1-12)$.

Solutions:

1. $4x+7\le 3x-2$

$x\le -9$

S. set $=\{x\,|\,x\le -9\}$.

2. $2x\le 2x-10$

$0\le -10$

We see that the variable x is disappeared and we found an inequality which is always false.

S. set $=\{\ \}$.

3. $x+3\ge 7-x$

$x+3\ge 7-x$

$2x \geq 4$

$x \geq 2$

S. set $= \{x \mid x \geq 2\}$.

4. $3x+1 < 5x-4$

$3x+1 < 5x-4$

$3x-5x < -4-1$

$-2x < -5$

$x > \frac{5}{2}$

S. set $= \left\{x \mid x > \frac{5}{2}\right\}$.

5. $15x+6 \geq 10x-24$

$15x+6 \geq 10x-24$

$15x-10x \geq -24-6$

$5x \geq -30$

$x \geq -6$

S. set $= \{x \mid x \geq -6\}$.

6. $9-7x > 19-2x$

$9-7x > 19-2x$

$-7x+2x > 19-9$

$-5x > 10$

$x < -\frac{10}{5}$

$x < -2$

S. set $= \{x \mid x < -2\}$.

7. $-2x+1 \leq x \leq 6-x$

Consider left inequality

$-2x+1 \leq x$

$-2x-x \leq -1$

$-3x \leq -1$

$x \geq \frac{1}{3}$ (1)

Consider right inequality

$x \leq 6-x$

$2x \leq 6$

$x \leq 3$ (2)

Combining (1) and (2), we have

$\frac{1}{3} \leq x \leq 3$

S. set $= \left\{x \mid \frac{1}{3} \leq x \leq 3\right\}$.

8. $2x-4 \leq x \leq 2x-10$

Consider left inequality

$2x-4 \leq x$

$x \leq 4$ (1)

Consider right inequality

$x \leq 2x-10$

$x-2x \leq -10$

$-x \leq -10$

$x \geq 10$ (2)

Combining (1) and (2), we have

$10 \leq x \leq 4$

S. set $= \{x \mid 10 \leq x \leq 4\}$.

9. $12 \geq x+16 \geq -20$

Subtracting 16 from each part

$12-16 \geq x \geq -20-16$

$-4 \geq x \geq -36$

S. set $= \{x \mid -4 \geq x \geq -36\}$.

10. $35 \leq 2x+5 \leq 80$

Subtracting 5 from each part

$35-5 \leq 2x \leq 80-5$

$30 \leq 2x \leq 75$

Dividing by 2

$15 \leq x \leq \frac{75}{2}$

S. set $= \left\{x \mid 15 \leq x \leq \frac{75}{2}\right\}$.

11. $0 \geq 20-x \geq -20$

Subtracting 20 from each part

$0-20 \geq -x \geq -20-20$

$-20 \geq -x \geq -40$

Multiply by x

$20 \leq x \leq 40$

S. set $= \{x \mid 20 \leq x \leq 40\}$.

12. $8+x \leq 2x-4 \leq 24$

Consider left inequality

$8+x \leq 2x-4$

$x-2x \leq -4-8$

$-x \leq -12$

$x \geq 12 \quad (1)$

Consider left inequality

$2x-4 \leq 24$

$2x \leq 28$

$x \leq 14 \quad (2)$

Combining (1) and (2), we have

$12 \leq x \leq 14$

S. set $= \{x \mid 12 \leq x \leq 14\}$.

In Exercises (13-20), Solve the following second-degree inequalities:

13. $x^2-5x+6 \leq 0$

The associated equation is

$x^2-5x+6=0$

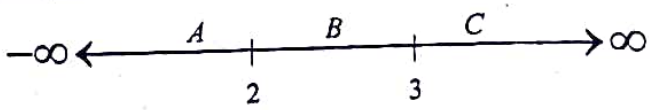
$x^2-3x-2x+6=0$

$x(x-3)-2(x-3)=0$

$(x-2)(x-3)=0$

$x=2 \quad , \quad x=3$

So $x=2,3$ are boundary numbers. We locate them on real line so that it is divided into regions A, B, C as shown:



For region A, test $x=1$: $(1)^2-5(1)+6 \leq 0 \quad 2 \leq 0$, False

For region B, test $x=2\cdot5$: $(2\cdot5)^2-5(2\cdot5)+6 \leq 0 \quad -0\cdot25 \leq 0$, True

For region C, test $x=4$: $(4)^2-5(4)+6 \leq 0 \quad 2 \geq 0$, True

Then the solution set is $[2,3] \cup [3,\infty)$ or $[2,\infty)$.

14. $x^2-2x-15>0$

The associated equation is

$x^2-2x-15=0$

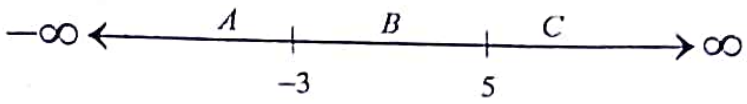
$x^2-5x+3x-15=0$

$x(x-5)+3(x-5)=0$

$(x+3)(x-5)=0$

$x=-3$, $x=5$

So $x=-3,5$ are boundary numbers. We locate them on real line so that it is divided into regions A,B,C as shown:



For region A, test $x=-4$: $(-4)^2-2(-4)-15>0$ $19>0$, True

For region B, test $x=0$: $(0)^2-2(0)-15>0$ $-15>0$, False

For region C, test $x=6$: $(6)^2-2(6)-15>0$ $9>0$, True

Then the solution set is $(-\infty,-3)\cup(5,\infty)$.

15. $x^2-16\le 0$

The associated equation is

$x^2-16=0$

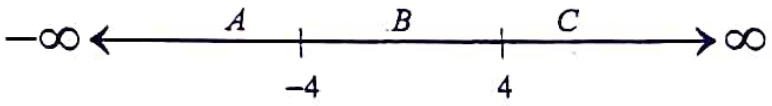
$(x)^2-(4)^2=0$

$(x-4)(x+4)=0$

$x+4=0$, $x-4=0$

$x=-4$, $x=4$

So $x=-4,4$ are boundary numbers. We locate them on real line so that it is divided into regions A,B,C as shown:



For region A, test $x=-5$: $(-5)^2-16\le 0$ $9\le 0$, False

For region B, test $x=0$: $(0)^2-16\le 0$ $-16\le 0$, True

For region C, test $x=5$: $(5)^2-16\le 0$ $9\le 0$, False

Then the solution set is $[-4,4]$.

16. $x^2-9\ge 0$

The associated equation is

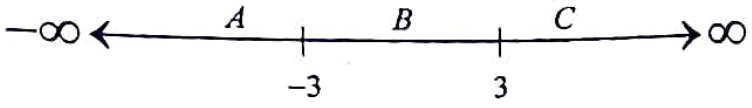
$x^2-9=0$

$(x)^2-(3)^2=0$

$(x-3)(x+3)=0$

$x=-3 \quad , \quad x=3$

So $x=-3,3$ are boundary numbers. We locate them on real line so that it is divided into regions A,B,C as shown:



For region A, test $x=-4$: $(-4)^2-9\geq 0 \quad 7\geq 0$, True

For region B, test $x=0$: $(0)^2-9\geq 0 \quad -9\geq 0$, False

For region C, test $x=4$: $(4)^2-9\geq 0 \quad 7\geq 0$, True

Then the solution set is $(-\infty,-3]\cup[3,\infty)$.

17. $x^2+2x-8\geq 0$

The associated equation is

$x^2+2x-8=0$

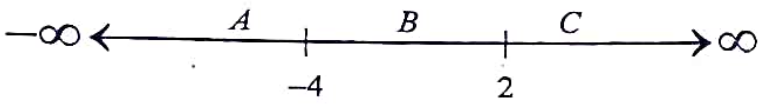
$x^2+4x-2x-8=0$

$x(x+4)-2(x+4)=0$

$(x+4)(x-2)=0$

$x=-4 \quad , \quad x=2$

So $x=-4,2$ are boundary numbers. We locate them on real line so that it is divided into regions A,B,C as shown:



For region A, test $x=-5$: $(-5)^2+2(-5)-8\geq 0 \quad 7\geq 0$, True

For region B, test $x=0$: $(0)^2+2(0)-8\geq 0 \quad -8\geq 0$, False

For region C, test $x=3$: $(3)^2+2(3)-8\geq 0 \quad 7\geq 0$, True

Then the solution set is $(-\infty,-4]\cup[2,\infty)$.

18. $x^2-2x-3\geq 0$

The associated equation is

$x^2-2x-3=0$

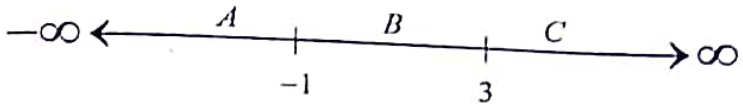
$x^2-3x+x-3=0$

$x(x-3)+1(x-3)=0$

$(x-3)(x+1)=0$

$x=3 \quad , \quad x=-1$

So $x=-1,3$ are boundary numbers. We locate them on real line so that it is divided into regions A,B,C as shown:



For region A, test $x=-2$: $(-2)^2-2(-2)-3\geq 0 \quad 5\geq 0$, True

For region B, test $x=0$: $(0)^2-2(0)-3\geq 0 \quad -3\geq 0$, False

For region C, test $x=4$: $(4)^2-2(4)-3\geq 0 \quad 5\geq 0$, True

Then the solution set is $(-\infty,-1]\cup[3,\infty)$.

19. $2x^2-x-10>0$

The associated equation is

$2x^2-x-10=0$

$2x^2-5x+4x-10=0$

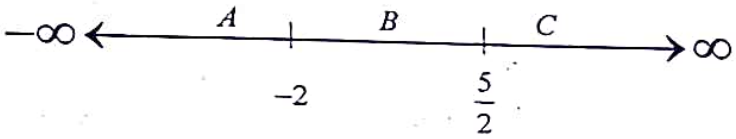
$x(2x-5)+2(2x-5)=0$

$(x+2)(2x-5)=0$

$x+2=0 \quad , \quad 2x-5=0$

$x=-2 \quad , \quad x=\frac{5}{2}$

So $x=-2,\frac{5}{2}$ are boundary numbers. We locate them on real line so that it is divided into regions A,B,C as shown:



For region A, test $x=-3$: $2(-3)^2-(-3)-10>0 \quad 11>0$, True

For region B, test $x=0$: $2(0)^2-(0)-10>0 \quad -10>0$, False

For region C, test $x=3$: $2(3)^2-(3)-10>0 \quad 5>0$, True

Then the solution set is $(-\infty,-2)\cup\left(\frac{5}{2},\infty\right)$.

20. $6x^2+x-12\geq 0$

The associated equation is

$6x^2+x-12=0$

$6x^2+9x-8x-12=0$

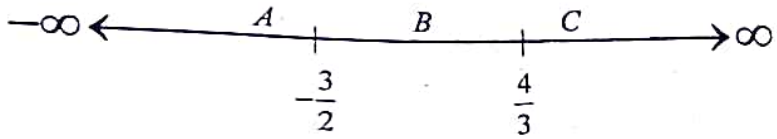
$3x(2x+3)-4(2x+3)=0$

$(2x+3)(3x-4)=0$

$2x=-3 \quad , \quad 3x=4$

$x=-\frac{3}{2} \quad , \quad x=\frac{4}{3}$

So $x=-\frac{3}{2},\frac{4}{3}$ are boundary numbers. We locate them on real line so that it is divided into regions A,B,C as shown:



For region A, test $x=-2$: $6(-2)^2+(-2)-12\geq 0 \quad 10\geq 0$, True

For region B, test $x=0$: $6(0)^2+(0)-12\geq 0 \quad -12\geq 0$, False

For region C, test $x=2$: $6(2)^2+(2)-12\geq 0 \quad 14\geq 0$,

Then the solution set is $\left(-\infty,-\frac{3}{2}\right]\cup\left[\frac{4}{3},\infty\right)$.

In Exercises (21-26), solve the following inequalities:

21. $|2x+5|>|2-5x|$

The associated equation is

$|2x+5|=|2-5x|$

$\pm(2x+5)=(2-5x)$

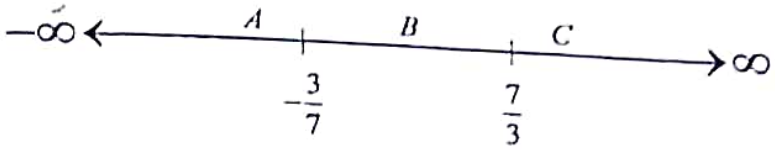
$2x+5=2-5x \quad$ or $\quad -(2x+5)=2-5x$

$2x+5x=2-5 \quad$ or $\quad -2x-5=2-5x$

$7x=-3 \quad$ or $\quad 3x=7$

$x=-\frac{3}{7} \qquad x=\frac{7}{3}$

So $x=-\frac{3}{7},\frac{7}{3}$ are boundary numbers. We locate them on real line so that it is divided into regions A,B,C as shown:



For region A, test $x=-1$: $|-2+5|>|2+5|$, False

For region B, test $x=0$: $|0+5|>|2-0|$, True

For region C, test $x=3$: $|6+5|>|2-15|$, False

Thus the solution set is $\left(-\frac{3}{7},\frac{7}{3}\right)$.

22. $\left|\frac{x+8}{12}\right|<\frac{x-1}{10}$

The associated equation is

$$\left|\frac{x+8}{12}\right|=\frac{x-1}{10}$$

$$\pm\left(\frac{x+8}{12}\right)=\frac{x-1}{10}$$

Multiplying by 60

$$\pm 5(x+8)=6(x-10)$$

$5(x+.8)=6x-10$ or $-5(x+8)=6x-10$

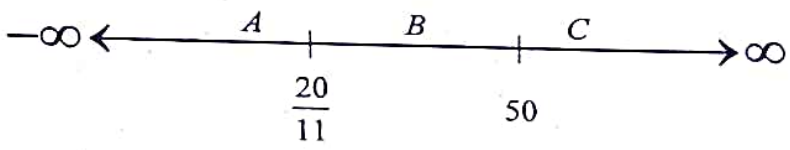
$5x+40=6x-10$ or $-5x-40=6x-60$

$40+10=6x-5x$ or $-5x-6x=-60+40$

$50=x$ or $-11x=-2$

$x=50$ or $x=\frac{20}{11}$

So $x=\frac{20}{11}$, 50 are boundary numbers. We locate them on real line so that it is divided into regions A,B,C as shown:



For region A, test $x=0$: $\left|\frac{0+8}{12}\right| < \frac{0-1}{10}$, False

For region B, test $x=10$: $\left|\frac{10+8}{12}\right| < \frac{10-1}{10}$, False

For region C, test $x=100$: $\left|\frac{100+8}{12}\right| < \frac{100-1}{10}$, True

Thus the solution set is $[50,\infty)$.

23. $\frac{x+1}{2} - \frac{1}{x} > \frac{4}{x} + 5$

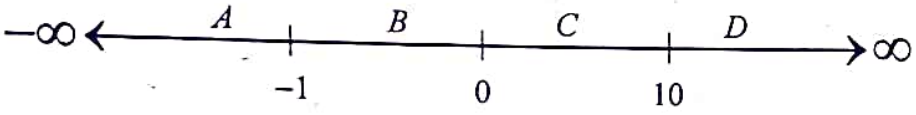
The associated equation is

$$\frac{x+1}{2} - \frac{1}{x} = \frac{4}{x} + 5$$

Multiplying both sides by $2x$

$$x(x+1) - 2 = 8 + 10x$$
$$x^2 + x - 2 - 8 - 10x = 0$$
$$x^2 - 9x - 10 = 0$$
$$x^2 - 10x + x - 10 = 0$$
$$x(x-10) + 1(x-10) = 0$$
$$(x+1)(x-10) = 0$$
$$x+1=0 \quad , \quad x-10=0$$
$$x=-1 \quad , \quad x=10$$

Also we see that the inequality is not defined at $x=0$. So $x=-1, 0, 10$ are boundary numbers. We locate them on real line so that it is divided into regions A, B, C, D as shown:



For region A, test $x=-2$: $\frac{-2+1}{2} - \left(\frac{1}{-2}\right) > \frac{4}{-2} + 5$, False

For region B, test $x=-\frac{1}{2}$: $\frac{-\frac{1}{2}+1}{2} - \left(\frac{1}{-\frac{1}{2}}\right) > \frac{4}{-\frac{1}{2}} + 5$

$$\Rightarrow \quad \frac{1}{4} + 2 > -8 + 5 \quad , \quad \text{True}$$

For region C, test $x=1$: $\frac{1+1}{2}-\left(\frac{1}{1}\right)>\frac{4}{1}+5$, False

For region D, test $x=11$: $\frac{11+1}{2}-\left(\frac{1}{11}\right)>\frac{4}{11}+5$, True

Thus the solution set is $(-1,0)\cup(10,\infty)$.

24. $x^{-2}-4x^{-1}+4>0$

$$\frac{1}{x^2}-\frac{4}{x}+4>0$$

The associated equation is

$$\frac{1}{x^2}-\frac{4}{x}+4=0$$

Multiplying both sides by x^2

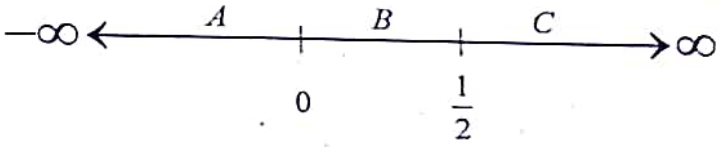
$$1-4x+4x^2=0$$

$$4x^2-4x+1=0$$

$$(2x-1)^2=0$$

$$2x-1=0 \Rightarrow x=\frac{1}{2}$$

Also we see that the inequality is not defined at $x=0$. So $x=0,\frac{1}{2}$ are boundary numbers. We locate them on real line so that it is divided into regions A,B,C as shown:



For region A, test $x=-1$: $\frac{1}{(-1)^2}-\frac{4}{(-1)}+4>0$, True

For region B, test $x=\frac{1}{4}$: $\frac{1}{\left(\frac{1}{4}\right)^2}-\frac{4}{\left(\frac{1}{4}\right)}+4>0$, True

For region C, test $x=1$: $\frac{1}{(1)^2}-\frac{4}{(1)}+4>0$, True

Thus the solution set is $(-\infty,0)\cup\left(0,\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$.

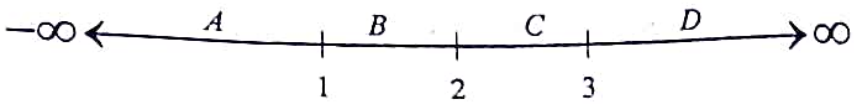
25. $\frac{x-2}{(x-1)(x-3)}\geq 0$

The associated equation is

$$\frac{x-2}{(x-1)(x-3)}=0$$

$$\Rightarrow \quad x-2=0 \Rightarrow \quad x=2$$

Also we see that the inequality is not defined at $x=1$ and $x=3$. So $x=1, 2, 3$ are boundary numbers. We locate them on real line so that it is divided into regions A, B, C, D as shown:



For region A, test $x=0$: $\quad \frac{0-2}{(0-1)(0-3)} \geq 0,$ False

For region B, test $x=\frac{1}{2}$: $\quad \frac{\frac{1}{2}-2}{\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-3\right)} \geq 0,$

$$\frac{-\frac{3}{2}}{\left(-\frac{1}{2}\right)\left(-\frac{5}{2}\right)} \geq 0, \quad \text{False}$$

For region C, test $x=\frac{5}{2}$: $\quad \frac{\frac{5}{2}-2}{\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-3\right)} \geq 0,$

$$\frac{\frac{1}{2}}{\left(\frac{3}{2}\right)\left(-\frac{1}{2}\right)} \geq 0, \quad \text{False}$$

For region D, test $x=4$: $\quad \frac{4-2}{(4-1)(4-3)} \geq 0,$ True

Thus the solution set is $(3,\infty)$.

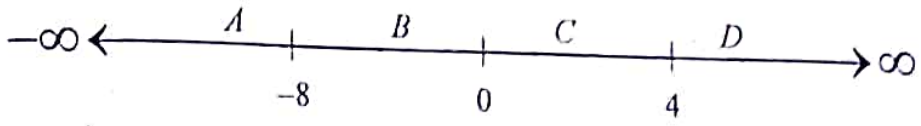
26. $\frac{2}{x} \leq \frac{3}{x-4}$

The associated equation is

$$\frac{2}{x}=\frac{3}{x-4}$$

$$2(x-4)=3x$$
$$2x-8=3x$$
$$3x-2x=-8$$
$$x=-8$$

Also we see that the inequality is not defined at $x=0$ and $x=4$.
So $x=-8, 0, 4$ are boundary numbers. We locate them on real line so that it is divided into regions A, B, C, D as shown:



For region A, test $x=-10$: $\frac{2}{-10}\leq\frac{3}{-10-4}$, False

For region B, test $x=-1$: $\frac{2}{-1}\leq\frac{3}{-1-4}$, True

For region C, test $x=1$: $\frac{2}{1}\leq\frac{3}{1-4}$, False

For region D, test $x=5$: $\frac{2}{5}\leq\frac{3}{5-4}$, True

Thus the solution set is $[-8,0)\cup(4,\infty)$.

27. Company A rent a car for Rs. 1500 a day and Rs. 15 for every km driven and company B rent a car for Rs. 2000 a day and Rs. 10 for every km driven. You want to rent a car for 3 days. How many km can you drive a company A car driving the 3 days if it is to cost less than a company B car?

Solution:

Let x be the number of Km.

Cost for rent a car A for 3 days $=15x+3(1500)$

Cost for rent a car B for 3 days $=10x+3(2000)$

According to condition

$$15x+3(1500)<10x+3(2000)$$
$$15x+4500<10x+6000$$
$$15x-10x<6000-4500$$
$$5x<1500$$
$$x<300$$

The cost of car A will be less than B if it is driven less than 300 km.

28. A café charges a Rs. 5000 flat fee for birthday party arrangements and Rs. 200 for each person. Natasha has no more than Rs. 12000 to spend on the birthday party. How many people can Natasha invite to her birthday party without exceeding her limit?

Solution:

Let x be the number of people invited by Natasha.

According to the condition

$$200x + 5000 \leq 12000$$

$$200x \leq 12000 - 5000$$

$$200x \leq 7000$$

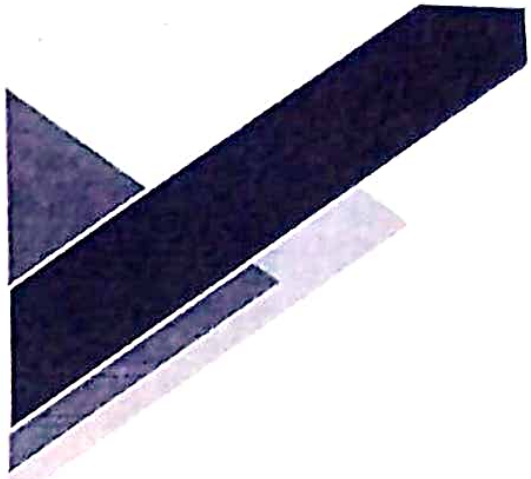
$$x \leq \frac{7000}{200}$$

$$x \leq 35$$

Natasha can invite 35 persons or less.

CHAPTER 2
FUNCTIONS AND THEIR GRAPHS

Functions and their graphs with examples



Exercises 2.1

In Exercises $1-10$, determine $f(0), f(-1), f(2)$ and $f(a+b)$.

1. $f(x)=2x-5$

$f(0)=2(0)-5=0-5=-5$

$f(-1)=2(-1)-5=-2-5=-7$

$f(2)=2(2)-5=4-5=-1$

$f(a+b)=2(a+b)-5$

2. $f(x)=7x+10$

$f(0)=7(0)+10=0+10=10$

$f(-1)=7(-1)+10=-7+10=3$

$f(2)=7(2)+10=14+10=24$

$f(a+b)=7(a+b)+10$

3. $f(x)=-x+4$

$f(0)=-(0)+4=0+4=4$

$f(-1)=-(-1)+4=1+4=5$

$f(2)=-2+4=2$

$f(a+b)=-(a+b)+4$

4. $f(x)=-\frac{x}{2}$

$f(0)=-\frac{0}{2}=0$

$f(-1)=-\frac{(-1)}{2}=\frac{1}{2}$

$f(2)=-\frac{2}{2}=-1$

$f(a+b)=-\frac{a+b}{2}$

5. $f(x)=x^2+3$

$f(0)=(0)^2+3=0+3=3$

$f(-1)=(-1)^2+3=1+3=4$

$f(2)=(2)^2+3=4+3=7$

$f(a+b)=(a+b)^2+3$

6. $f(u)=u^2+\frac{u}{2}+6$

$f(0)=(0)^2+\frac{0}{2}+6=6$

$f(-1)=(-1)^2+\frac{(-1)}{2}+6$

$$=1-\frac{1}{2}+6$$

$$=\frac{2-1+12}{2}=\frac{13}{2}$$

$$f(2)=(2)^2+\frac{2}{2}+6=11$$

$$f(a+b)=(a+b)^2+\frac{a+b}{2}+6$$

$$=\frac{2(a+b)^2+a+b+12}{2}$$

7. $f(t)=t^2-2t+1$

$$f(0)=(0)^2-2(0)+1=1$$

$$f(-1)=(-1)^2-2(-1)+1=1+2+1=4$$

$$f(2)=(2)^2-2(2)+1=4-4+1=1$$

$$f(a+b)=(a+b)^2-2(a+b)+1$$

8. $f(n)=n^2$

$$f(0)=(0)^2=0$$

$$f(-1)=(-1)^2=1$$

$$f(2)=(2)^2=4$$

$$f(a+b)=(a+b)^2$$

9. $f(x)=200$

$$f(0)=200$$

$$f(-1)=200$$

$$f(2)=200$$

$$f(a+b)=200$$

10. $f(x)=25-\frac{x^2}{2}$

$$f(0)=25-\frac{(0)^2}{2}=25-0=25$$

$$f(-1)=25-\frac{(-1)^2}{2}=25-\frac{1}{2}=\frac{50-1}{2}=\frac{49}{2}$$

$$f(2)=25-\frac{(2)^2}{2}=25-\frac{4}{2}=25-2=23$$

$$f(a+b)=25-\frac{(a+b)^2}{2}$$

In Exercises $11-14$, using $f(x)$, simplify the difference quotient $\left(\frac{f(x+h)-f(x)}{h},\ h\neq 0\right)$.

11. $f(x)=2x+3$

The difference quotient is

$$=\frac{f(x+h)-f(x)}{h}$$

$$=\frac{[2(x+h)+3]-[2x+3]}{h}$$

$$=\frac{2x+2h+3-2x-3}{h}=\frac{2h}{h}=2$$

12. $f(x)=3-4x$

The difference quotient is

$$=\frac{f(x+h)-f(x)}{h}$$

$$=\frac{[3-4(x+h)]-[3-4x]}{h}$$

$$=\frac{3-4x-4h-3+4x}{h}$$

$$=-\frac{4h}{h}=-4$$

13. $f(x)=x^2+6x-4$

The difference quotient is

$$=\frac{f(x+h)-f(x)}{h}$$

$$=\frac{[(x+h)^2+6(x+h)-4]-[x^2+6x-4]}{h}$$

$$= \frac{x^2 + 2xh + h^2 + 6x + 6h - 4 - x^2 - 6x + 4}{h} = \frac{2xh + h^2 + 6h}{h}$$

$$= \frac{h(2x + h + 6)}{h} = 2x + h + 6$$

14. $f(x) = 2x^2 + 3x - 7$

The difference quotient is

$$= \frac{f(x+h) - f(x)}{h}$$

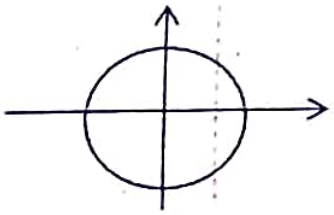
$$= \frac{\left[2(x+h)^2 + 3(x+h) - 7\right] - \left[2x^2 + 3x - 7\right]}{h}$$

$$= \frac{\left[2(x^2 + 2hx + h^2) + 3(x+h) - 7\right] - \left[2x^2 + 3x - 7\right]}{h} = \frac{2x^2 + 4hx + 2h^2 + 3x + 3h - 7 - 2x^2 - 3x + 7}{h}$$

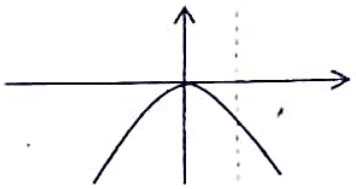
$$= \frac{4hx + 2h^2 + 3h}{h} = \frac{h(4x + 2h + 3)}{h} = 4x + 2h + 3$$

15. Use the vertical line test to determine whether the relation is a function or not.

a.
The vertical line intersects the graph at two points, so the given graph is not a function.

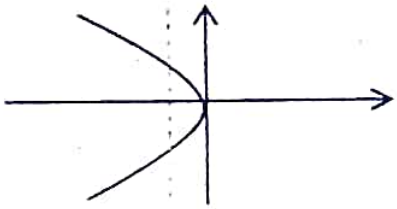


b.
The vertical line intersects the graph at one point only, so the given graph is a function.



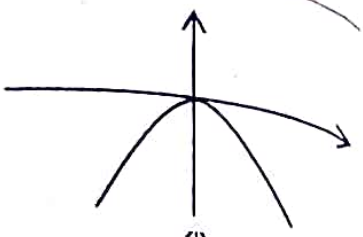
c.
The vertical line intersects the graph at one point only, so the given graph is a function.

d. The vertical line intersects the graphs at two points, so the given graph is not a function.



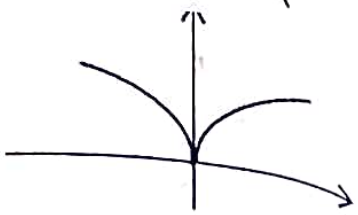
16. Classify the function as even or odd.

a.



Since the graph is symmetric about $y-axis$, therefore it is an even function.

b.

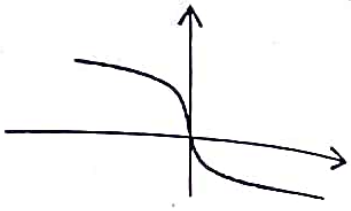


Since the graph is symmetric about $y-axis$, therefore it is an even function.

c.

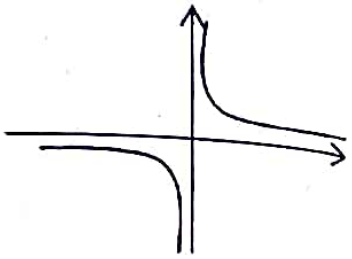
Since the graph is neither symmetric about $y-axis$ nor symmetric about origin, so it is neither even nor odd function.

d.



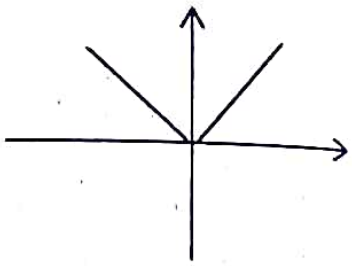
Since the graph is symmetric about origin, therefore it is an odd function.

e.



Since the graph is symmetric about origin, therefore it is an odd function.

f.



Since the graph is symmetric about $y-axis$, therefore it is an even function.

In Exercises $(17-22)$, determine whether the function is even, odd, or neither.

17. $f(x)=2x^2+4$

$f(-x)=2(-x)^2+4$

$=2x^2+4=f(x)$

So $f(-x)=f(x)$, therefore f is an even function.

18. $f(x)=x^3+x$

$f(-x)=(-x)^3+(-x)=-x^3-x$

$=-(x^3+x)=-f(x)$

So $f(-x)=-f(x)$, therefore f is an odd function.

[closed interval (open interval

. $f(x)=\sqrt{x}$

$f(-x)=\sqrt{-x}=\sqrt{x}\,i$

Since $f(-x)\neq f(x)$ and $f(-x)\neq -f(x)$, therefore f is neither even nor odd.

. $f(x)=x\sqrt{x^2+5}$

$f(-x)=-x\sqrt{(-x)^2+5}$

$=-x\sqrt{x^2+5}=-f(x)$

Since $f(-x)=-f(x)$, therefore f is an odd function.

. $f(x)=\dfrac{3x}{x^2+1}$

$f(-x)=\dfrac{3(-x)}{(-x)^2+1}$

$=-\dfrac{3(x)}{x^2+1}=-f(x)$

Since $f(-x)=-f(x)$, therefore f is an odd function.

2. $f(x)=(x+1)^2$

$f(-x)=(-x+1)^2$

Since $f(-x)\neq f(x)$ and $f(-x)\neq -f(x)$, therefore f is neither even nor odd.

n Exercises, $(23-30)$. Determine the omain.

3. $y=\sqrt{x+4}$

The quantity $x+4$ cannot be negative, so

$x+4\geq 0$

$x\geq -4$ or $[-4,\infty)$

Hence domain of $y=\sqrt{x+4}$ is $[-4,\infty)$.

24. $y=\sqrt{x^2-9}$

The quantity x^2-9 cannot be negative, so

$x^2-9\geq 0$

$x^2\geq 9$

Taking square root

$\pm x\geq 3$

$-x\geq 3$ and $x\geq 3$

$x\leq -3$ and $x\geq 3$

or $(-\infty,-3]\cup[3,\infty)$

Hence domain of $y=\sqrt{x^2-9}$ is $(-\infty,-3]\cup[3,\infty)$.

25. $y=-2+\sqrt{1-x}$

The quantity $\sqrt{1-x}$ can not be negative, so

$1-x\geq 0$

$1\geq x$ or $x\leq 1$ or $(-\infty,1]$

Hence domain of $y=-2+\sqrt{1-x}$ is $(-\infty,1]$.

26. $y=|x|-2$

Clearly domain of $y=|x|-2$ is $(-\infty,\infty)$.

27. $y=|x-3|$

Clearly domain of $y=|x-3|$ is $(-\infty,\infty)$.

28. $y=\dfrac{x+1}{x^3-4x}$

The quantity x^3-4x cannot be zero.

Take $x^3-4x=0$ $x\neq 0$

$x(x^2-4)\neq 0$ $x-2\neq 0$

$x(x-2)(x+2)\neq 0$ $x+2\neq 0$

$x\neq -2,0,2$

We see that the domain of

$y=\dfrac{x+1}{x^3-4x}$ is

$(-\infty,\infty)-\{-2,0,2\}$ or $R-\{-2,0,2\}$.

29. $y=\dfrac{2x}{6x^2+13x-5}$

The quantity $6x^2+13x-5$ cannot be zero.

Take $6x^2+13x-5=0$

$6x^2+15x-2x-5=0$

$3x(2x+5)-1(2x+5)=0$

$(3x-1)(2x+5)=0$

$3x-1\neq 0$, $2x+5\neq 0$

$x\neq \dfrac{1}{3}$, $x\neq \dfrac{-5}{2}$

We see that the domain of

$y=\dfrac{2x}{6x^2+13x-5}$ is

$(-\infty,\infty)-\left\{\dfrac{1}{3},\dfrac{-5}{2}\right\}$ or $R-\left\{\dfrac{1}{3},\dfrac{-5}{2}\right\}$.

30. $y=\dfrac{\sqrt{x-1}}{x^2-4}$

The quantity x^2-4 cannot be zero and $x-1$ cannot be negative, so

Take $x^2-4=0$

$x=\pm 2$

$\Rightarrow$ $x=-2,\ 2$ are not included in domain.

Also $x-1\geq 0 \Rightarrow x\geq 1$

Hence domain of $y=\dfrac{\sqrt{x-1}}{x^2-4}$ is

$[1,\infty)-\{2\}$ OR $[1,2)\cup(2,\infty)$.

In Exercises, $(31-33)$, Find $f+g$ and $f.g$. Also find their domain.

31. $f(x)=x,\ g(x)=\sqrt{x-1}$

$(f+g)(x)=f(x)+g(x)=x+\sqrt{x-1}$

$(f.g)(x)=f(x).g(x)=x\sqrt{x-1}$

We see that

$\text{Dom } f=(-\infty,\infty)$, $\text{Dom } g=[1,\infty)$

Domain of $f+g$ and $f.g$ is

$\text{Dom } f\cap \text{Dom } g=(-\infty,\infty)\cap[1,\infty)$

$=[1,\infty)$

32. $f(x)=\sqrt{x+1},\ g(x)=\sqrt{x-1}$

$(f+g)(x)=f(x)+g(x)$

$=\sqrt{x+1}+\sqrt{x-1}$

$(f.g)(x)=f(x).g(x)$

$=\sqrt{x+1}.\sqrt{x-1}=\sqrt{x^2-1}$

We see that

$\text{Dom } f=[-1,\infty)$, $\text{Dom } g=[1,\infty)$

Domain of $f+g$ and $f.g$ is

$\text{Dom } f\cap \text{Dom } g=[-1,\infty)\cap[1,\infty)$

$=[1,\infty)$

33. $f(x)=\dfrac{2}{x+1},\ g(x)=\dfrac{x}{x-4}$

$(f+g)(x)=f(x)+g(x)$

$=\dfrac{2}{x+1}+\dfrac{x}{x-4}$

$=\dfrac{2(x-4)+x(x+1)}{(x+1)(x-4)}$

$=\dfrac{2x-8+x^2+x}{(x+1)(x-4)}$

$=\dfrac{x^2+3x-8}{(x+1)(x-4)}$

$(f.g)(x)=f(x).g(x)$

$=\dfrac{2}{x+1}.\dfrac{x}{x-4}$

$=\dfrac{2x}{(x+1)(x-4)}$

We see that

$\text{Dom } f=R-\{-1\}$, $\text{Dom } g=R-\{4\}$

Domain of $f+g$ and $f.g$ is

$\text{Dom } f \cap \text{Dom } g = R - \{-1, 4\}$

Exercises, (34 – 36). Find the domain $\frac{f}{g}$ and $\frac{g}{f}$.

. $f(x) = 4, \quad g(x) = x^2 + 1$

$\text{Dom } f = (-\infty, \infty)$, $\text{Dom } g = (-\infty, \infty)$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{4}{x^2+1}$$

$$\text{Dom } \frac{f}{g} = (-\infty, \infty) \quad \text{or} \quad R$$

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{x^2+1}{4}$$

$$\text{Dom } \frac{g}{f} = (-\infty, \infty) \quad \text{or} \quad R$$

5. $f(x) = 2, \quad g(x) = 1 + \sqrt{x}$

$\text{Dom } f = (-\infty, \infty)$, $\text{Dom } g = [0, \infty)$

The quantity x should be non-negative, so $x \geq 0$ or $[0, \infty)$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2}{1+\sqrt{x}}$$

$$\text{Dom } \frac{f}{g} = [0, \infty)$$

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{1+\sqrt{x}}{2}$$

$$\text{Dom } \frac{g}{f} = [0, \infty)$$

36. $f(x) = \frac{2x}{x-4}, g(x) = \frac{x}{x+5}$

$\text{Dom } f = R - \{4\}$, $\text{Dom } g = R - \{-5\}$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\frac{2x}{x-4}}{\frac{x}{x+5}}$$

$$= \frac{2x}{x-4} \cdot \frac{x+5}{x} = \frac{2(x+5)}{x-4}$$

$$Dom \frac{f}{g} = Dom\, f \cap Dom\, g = R - \{4, -5\}$$

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\frac{x}{x+5}}{\frac{2x}{x-4}}$$

$$= \frac{x}{x+5} \cdot \frac{x-4}{2x} = \frac{x-4}{2(x+5)}$$

$$Dom \frac{g}{f} = Dom\, f \cap Dom\, g = R - \{4, -5\}$$

Exercises 2.2

In Exercises, (1 – 4.) Find $f^{-1}(x)$ and $f^{-1}(-1)$.

1. $f(x) = 2x + 3$

Let $y = 2x + 3$

$y - 3 = 2x$ or $2x = y - 3$

$$x = \frac{y-3}{2}$$

$$f^{-1}(y) = \frac{y-3}{2}$$

Replace y by x

$$f^{-1}(x) = \frac{x-3}{2}$$

Also,

$$f^{-1}(-1) = \frac{-1-3}{2} = \frac{-4}{2} = -2$$

2. $f(x) = -2x + 8$

Let $y = -2x + 8$

$$2x = 8 - y \Rightarrow x = \frac{8-y}{2}$$

$$f^{-1}(y)=\frac{8-y}{2}$$

Replacing y by x

$$f^{-1}(x)=\frac{8-x}{2}$$

Also, $f^{-1}(-1)=\frac{8-(-1)}{2}=\frac{9}{2}$

3. $f(x)=2x+1$

Let $y=2x+1$

$2x=y-1$

$$x=\frac{y-1}{2}$$

$$f^{-1}(y)=\frac{y-1}{2}$$

Replacing y by x

$$f^{-1}(x)=\frac{x-1}{2}$$

Also,

$$f^{-1}(-1)=\frac{-1-1}{2}=\frac{-2}{2}=-1$$

4. $f(x)=\frac{2x+1}{x-1}$, $x>1$

Let $y=\frac{2x+1}{x-1}$

$y(x-1)=2x+1$

$xy-y=2x+1$

$xy-2x=y+1$

$x(y-2)=y+1$

$$x=\frac{y+1}{y-2}$$

$$f^{-1}(y)=\frac{y+1}{y-2}$$

Replacing y by x

$$f^{-1}(x)=\frac{x+1}{x-2}$$

Also,

$$f^{-1}(-1)=\frac{-1+1}{-1-2}=\frac{0}{-3}=0$$

In Exercises, (5 – 8), .Find $(fog)(x)$ and $(gof)(x)$.

5. $f(x)=2x+1$, $g(x)=\frac{3}{x-1}$, $x\neq 1$

$$(fog)(x)=f(g(x))$$

$$=2g(x)+1=2\left(\frac{3}{x-1}\right)+1$$

$$=\frac{6}{x-1}+1=\frac{6+x-1}{x-1}$$

$$=\frac{x+5}{x-1}$$

$$(gof)(x)=g(f(x))=\frac{3}{f(x)-1}$$

$$=\frac{3}{2x+1-1}=\frac{3}{2x}$$

6. $f(x)=\sqrt{x+1}$, $g(x)=\frac{1}{x^2}$, $x\neq 0$

$$(fog)(x)=f(g(x))$$

$$=\sqrt{g(x)+1}$$

$$=\sqrt{\frac{1}{x^2}+1}=\sqrt{\frac{1+x^2}{x^2}}$$

$$(gof)(x)=g(f(x))=\frac{1}{(f(x))^2}$$

$$=\frac{1}{\left(\sqrt{x+1}\right)^2}=\frac{1}{x+1}$$

7. $f(x)=\sqrt{x-15}$, $g(x)=x^2+2x$

$$(fog)(x)=f(g(x))$$

$$=\sqrt{g(x)-15}$$

$$=\sqrt{x^2+2x-15}$$

$$(gof)(x)=g(f(x))=(f(x))^2+2f(x)$$

$$=\left(\sqrt{x-15}\right)^2+2\sqrt{x-15}$$

$$=x-15+2\sqrt{x-15}$$

8. $f(x)=\sqrt{25-x^2}$, $g(x)=\sqrt{x-3}$

$$(f\,o\,g)(x)=f(g(x))$$

$$=\sqrt{25-(g(x))^2}$$

$$=\sqrt{25-\left(\sqrt{x-3}\right)^2}$$

$$=\sqrt{25-x+3}$$

$$=\sqrt{28-x}$$

$$(g\,o\,f)(x)=g(f(x))$$

$$=\sqrt{f(x)-3}\quad=\sqrt{\sqrt{25-x^2}-3}$$

In Exercises, $(9-10,)$ if $A=\{1,2,3,4\}, B=\{5,6,7\}$ determine whether the function is onto or into?

9. $f(1)=5$, $f(2)=7$

$f(3)=7$, $f(4)=6$

We can easily check from arrow diagram.

Since Range $f=\{5,6,7\}=B$

therefore f is onto.

10. $f(3)=5$, $f(1)=5$

$f(2)=6$, $f(4)=6$

We can easily check from arrow diagram.

Since Range $f=\{5,6\}\neq B$

therefore f is an into function.

In Exercises $(11-20)$, identify as ceiling or floor functions and find their value.

11. $\lceil 3\cdot 7\rceil$, Ceiling function

$\lceil 3\cdot 7\rceil=4$

12. $\lfloor 2\cdot 11\rfloor$, Floor function

$\lfloor 2\cdot 11\rfloor=2$

13. $[-0\cdot 9]$, Floor function

$[-0\cdot 9]=-1$

14. $\lceil 8\rceil$, Ceiling function

$\lceil 8\rceil=8$

15. $\lceil 0\cdot 1\rceil$, Ceiling function

$\lceil 0\cdot 1\rceil=1$

16. $\lfloor 4\cdot 01\rfloor$, Floor function

$\lfloor 4\cdot 01\rfloor=4$

17. $\lceil 2\cdot 3\rceil+\lfloor -2\cdot 3\rfloor$, Ceiling and floor functions

$\lceil 2\cdot 3\rceil+\lfloor -2\cdot 3\rfloor=3+(-3)=0$

18. $\lfloor 3\cdot 5+\sqrt{2}\rfloor$, Floor function

$$\lfloor 3\cdot 5+\sqrt{2}\rfloor=\lfloor 3\cdot 5+1\cdot 41\rfloor$$

$$=\lfloor 4\cdot 91\rfloor=4$$

19. $\lfloor\lceil 2\cdot 5\rceil\rfloor$, Floor and Ceiling functions

$\lfloor\lceil 2\cdot 5\rceil\rfloor=\lfloor 3\rfloor=3$

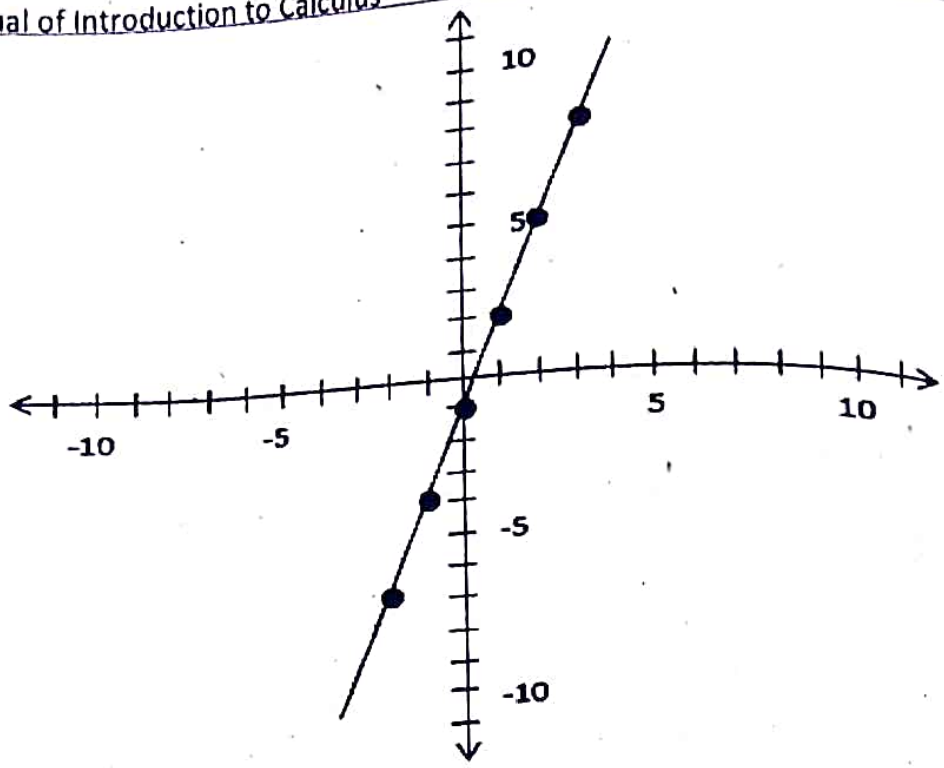
20. $\left[\frac{1}{100}\right]$, Floor function

$[0\cdot 01]=0$

Exercises 2.3

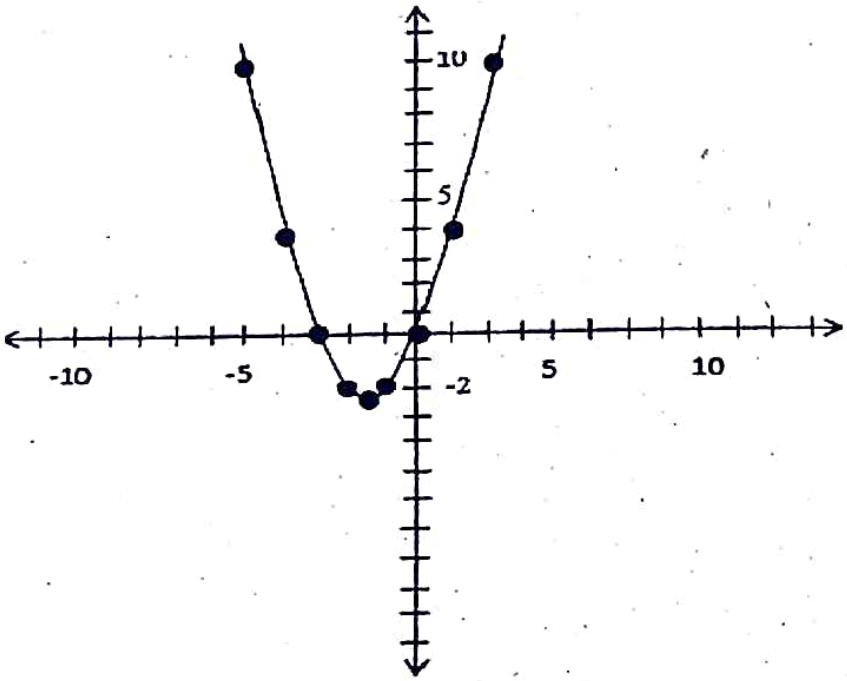
1. Sketch the graph of linear function for given sample point.

x	0	1	2	3	-1	-2
y	-1	2	5	8	-4	-7



2. Plot the graph of quadratic function for given sample points.

x	0	1	2	-1	- 1.5	- 2	-3	-4	-5
y	0	4	10	-2	- 2.25	-2	0	4	10

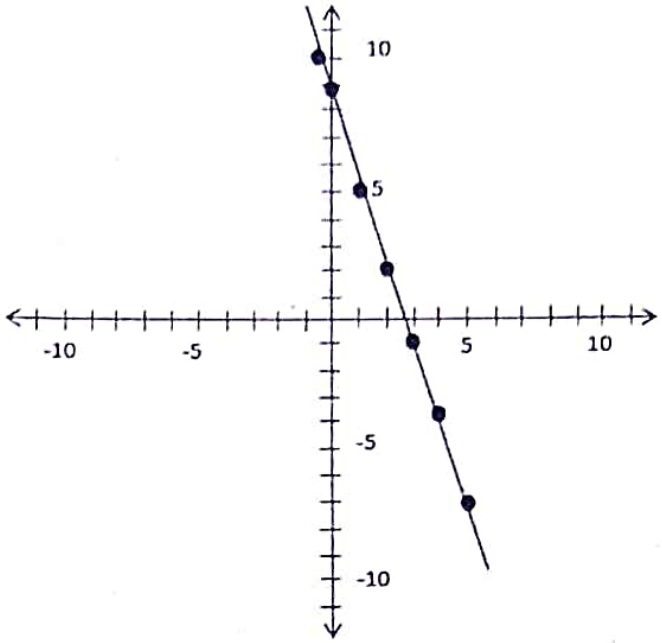


In Exercises, (3 – 8), Sketch the graph of given functions.

3. $f(x) = 8 - 3x$

Some sample points can be obtained by setting different values of x as listed in table .

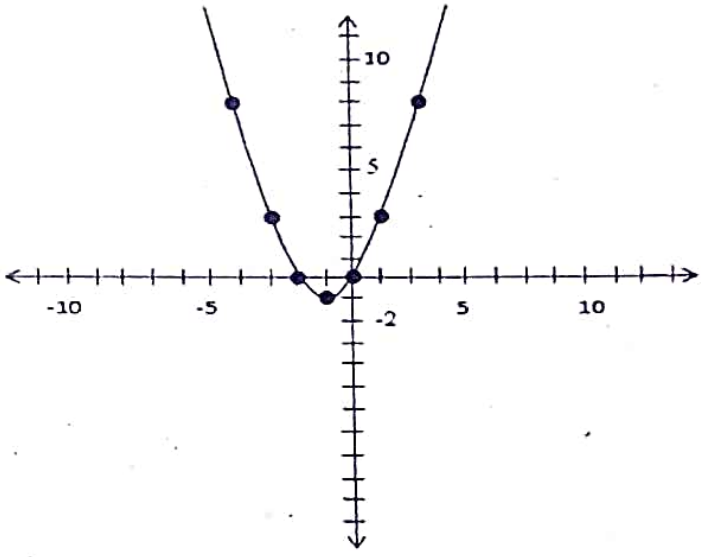
x	0	1	2	3	4	5	-1
y	8	5	2	-1	- 4	-7	11



4. $f(x) = x^2 + 2x$

Some sample points can be obtained by setting different values of x as listed in table .

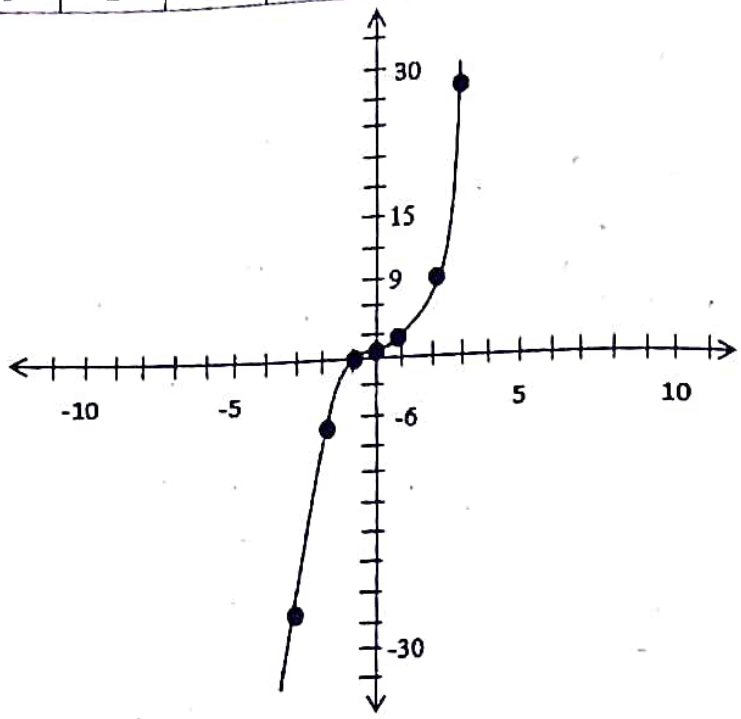
x	0	1	2	-1	-2	- 3	-4
y	0	3	8	-1	0	3	8



5. $f(x)=x^3+1$

Some sample points can be obtained by setting different values of x as listed in table.

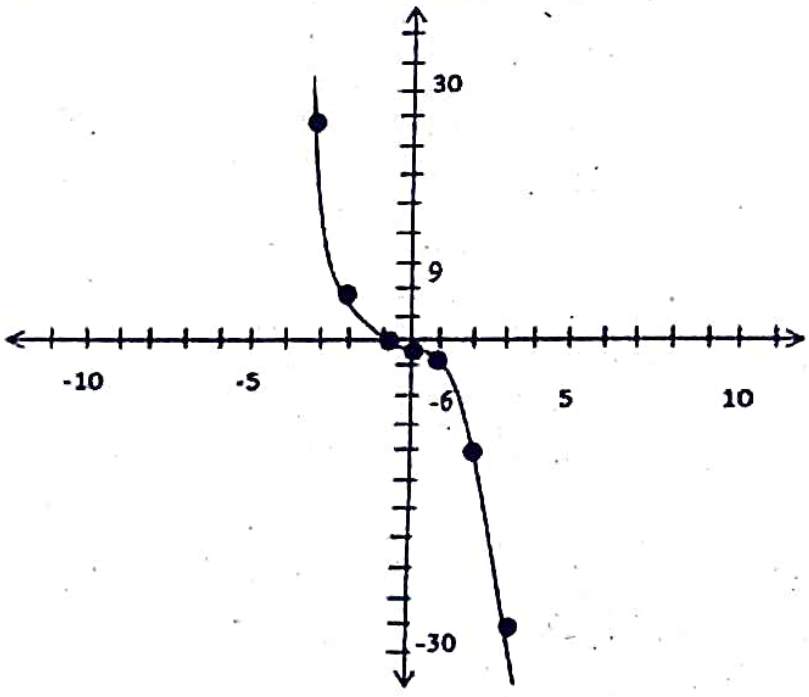
x	0	1	2	3	-1	- 2	-3
y	1	2	9	28	0	-7	-26



6. $f(x)=-x^3-1$

Some sample points can be obtained by setting different values of x as listed in table .

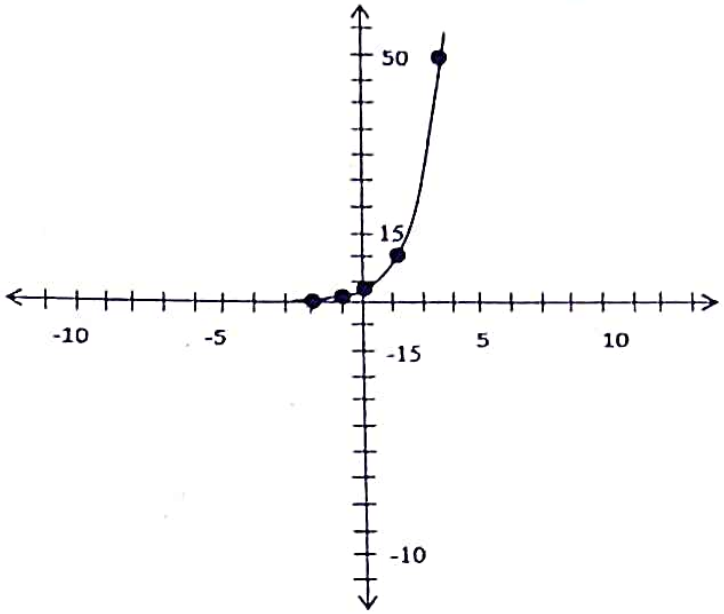
x	0	1	2	3	-1	-2	- 3
y	-1	-2	-9	-28	0	7	26



7. $f(x)=2\cdot 5^x$

Some sample points can be obtained by setting different values of x as listed in table.

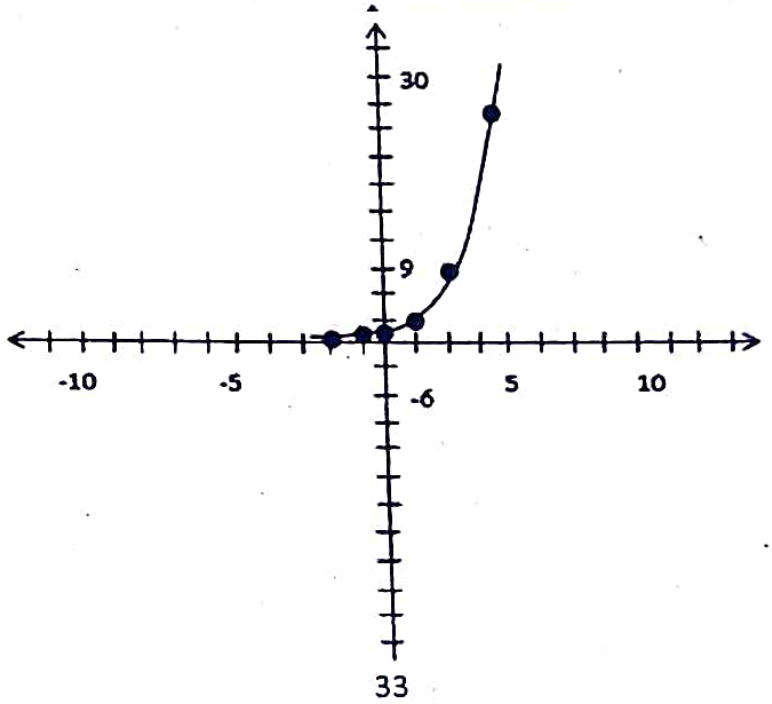
x	0	1	2	-1	-2
y	2	10	50	0.4	0.08



8. $f(x)=3^x$

Some sample points can be obtained by setting different values of x as listed in table .

x	0	1	2	3	-1	-2
y	1	3	9	27	0.33	0.11

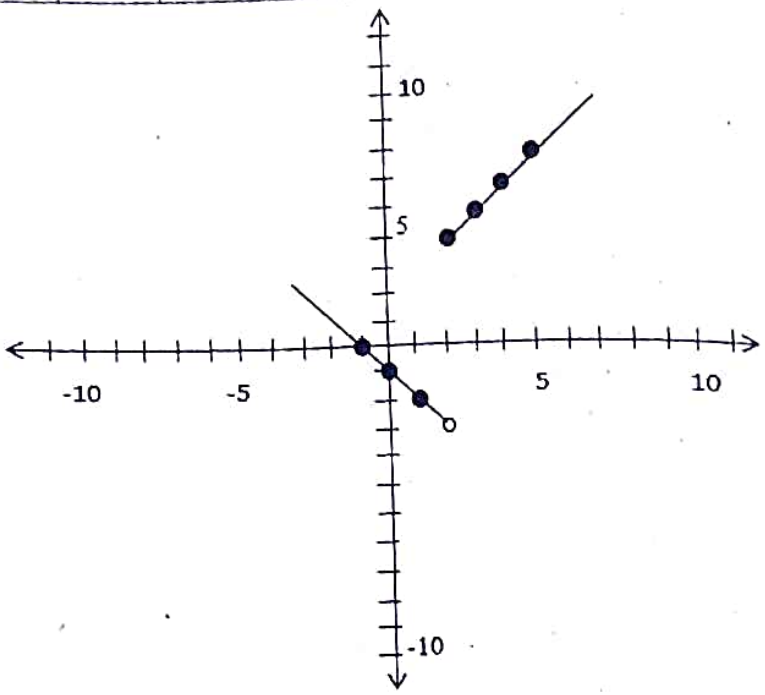


In Exercises (9 – 12), Sketch the graph of piecewise function.

9. $f(x)=\begin{cases} x+2 & \text{if } x \geq 2 \\ -x-1 & \text{if } x<2 \end{cases}$

Some sample points can be obtained by setting different values of x as listed in table.

x	2	3	4	5	1	0	-1
y	4	5	6	7	-2	-1	0



10. $f(x)=\begin{cases} x^2 & \text{if } x \geq 0 \\ -x-2 & \text{if } x<0 \end{cases}$

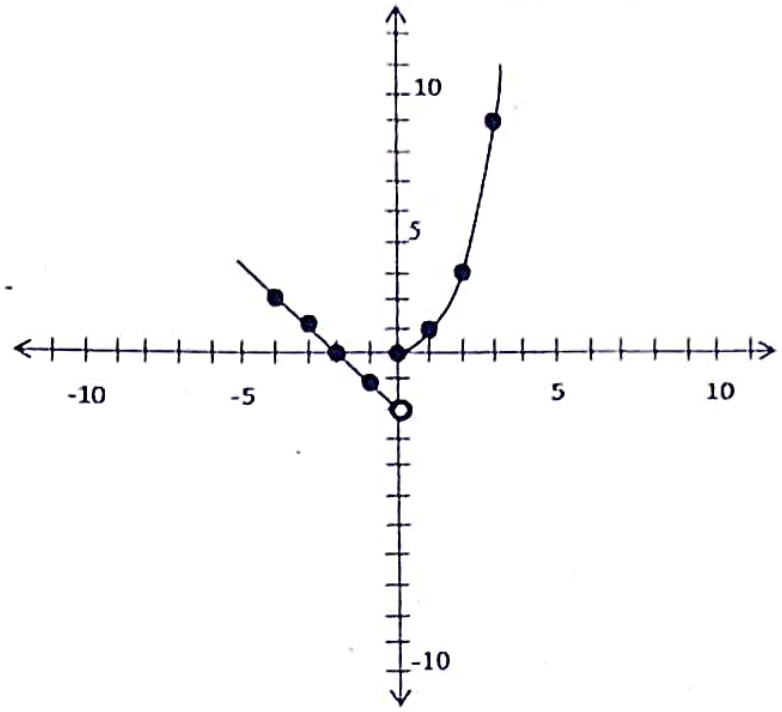
The given function is piecewise linear and quadratic function. Some sample points can be listed in table.

For $f(x)=x^2 \;;\; x \geq 0$

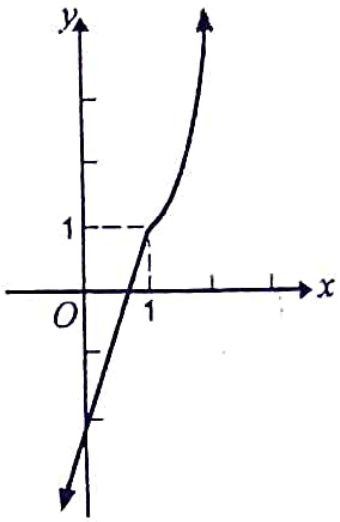
x	0	1	2	3
y	0	1	4	9

For $f(x)=-x-2 \;,\; x<0$

x	-1	-2	-3	-4
y	-1	0	1	2



11. $g(x)=\begin{cases}3x-2 & \text{if } x<1\\ x^2 & \text{if } x\geq 1\end{cases}$



The given function is piecewise linear and quadratic function. Some sample points can be listed in table.

For $f(x)=x^2$; $x\geq 1$

x	0	1	2
y	0	1	4

For $f(x)=3x-2$, $x<1$

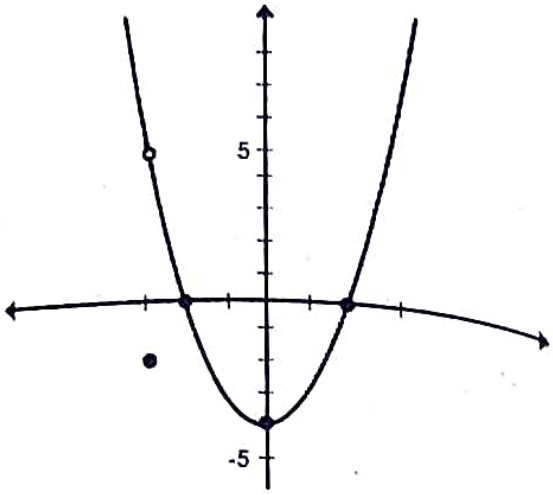
x	0.75	0.5	0.25	0
y	0.25	-0.5	-1.25	-2

12. $h(x)=\begin{cases}x^2-4 & \text{if } x\neq -3\\ -2 & \text{if } x=-3\end{cases}$

The given function is piecewise linear and quadratic function. Some sample points can be listed in table.

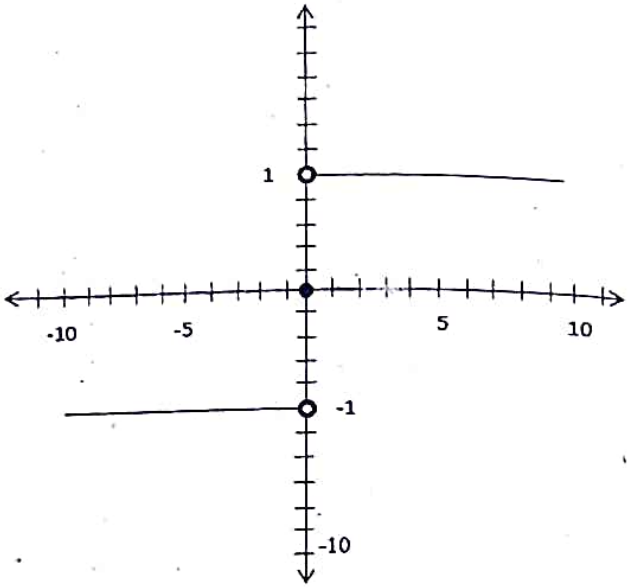
For $f(x)=x^2-4 \;;\; x\neq 3$

x	0	2	3	-2	-3
y	-4	0	5	0	5



13. Sketch the signum (**sgm**) function.

$$f(x)=\begin{cases}1 & \text{if } x>0\\ 0 & \text{if } x=0\\ -1 & \text{if } x<0\end{cases}$$

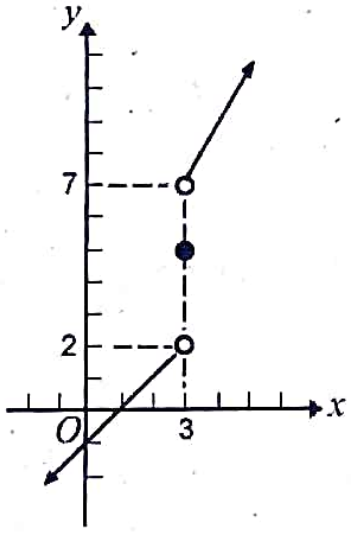


14. Let f be a function defined by

$$f(x)=\begin{cases}x-1 & \text{if } x<3\\ 5 & \text{if } x=3\\ 2x+1 & \text{if } 3<x\end{cases}$$

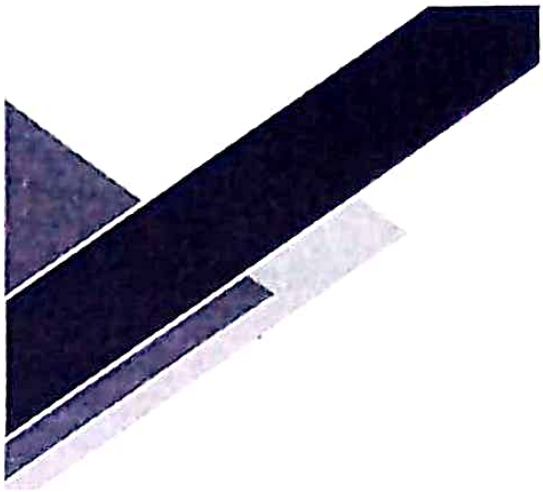
Some sample points can be obtained by setting different values of x as listed in table.

x	0	1	2	3	4	5
y	-1	0	1	5	9	11



CHAPTER 3
LIMIT AND CONTINUITY

Functions and their graphs with examples



Exercises 3.1

In Exercises (1-10), find the limits.

Solutions:

1. $\lim_{x\to 2}(2x-1)=2(2)-1=4-1=3$

2. $\lim_{x\to -2}(x^3+10)=(-2)^3+10=-8+10=2$

3. $\lim_{x\to 2}(-x^2)=-(2)^2=-4$

4.

$$\lim_{x\to 0}\frac{2x+5}{x+1}=\frac{\lim_{x\to 0}(2x+5)}{\lim_{x\to 0}(x+1)}$$

$$=\frac{2(0)+5}{0+1}=\frac{0+5}{0+1}=5$$

5. $\lim_{t\to -1}(8-3t)(2t-1)$

$$=\lim_{t\to -1}(8-3t).\lim_{t\to -1}(2t-1)$$

$$=[8-3(-1)][2(-1)-1]$$

$$=(8+3)(-2-1)=(11)(-3)=-33$$

6. $\lim_{x\to 7}7=7$

7.

$$\lim_{t\to 5}10(t-8)(t-12)$$

$$=\lim_{t\to 5}10.\lim_{t\to 5}(t-8).\lim_{t\to 5}(t-12)$$

$$=10(5-8)(5-12)=10(-3)(-7)=210$$

8. $\lim_{x\to 5}\frac{x+2}{x-4}=\frac{\lim_{x\to 5}(x+2)}{\lim_{x\to 5}(x-4)}=\frac{5+2}{5-4}=7$

9. $\lim_{h\to 2}\frac{h+4}{2h+1}=\frac{\lim_{h\to 2}(h+4)}{\lim_{h\to 2}(2h+1)}=\frac{2+4}{2(2)+1}=\frac{6}{5}$

10.

$$\lim_{x\to 3}\frac{x^2-2x}{x+1}=\frac{\lim_{x\to 3}(x^2-2x)}{\lim_{x\to 3}(x+1)}$$

$$=\frac{(3)^2-2(3)}{3+1}=\frac{9-6}{4}=\frac{3}{4}$$

In Exercise (11-20), use an algebraic simplification to help find the limit, if it exists.

11.

$$\lim_{x\to -1}\frac{(x+1)(x^2+3)}{x+1}, \quad \left(\frac{0}{0}\right) \textit{ form}$$

$$=\lim_{x\to -1}(x^2+3)=(-1)^2+3=1+3=4$$

12.

$$\lim_{x\to 2}\frac{x^2-4}{x-2}, \quad \left(\frac{0}{0}\right) \textit{ form}$$

$$=\lim_{x\to 2}\frac{(x)^2-(2)^2}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}$$

$$= \lim_{x\to 2}(x+2) = 2+2 = 4$$

13.

$$\lim_{t\to -2}\frac{t^3+8}{t+2} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{t\to -2}\frac{(t)^3+(2)^3}{t+2} = \lim_{t\to -2}\frac{(t+2)(t^2+2t+4)}{t+2}$$

$$= \lim_{t\to -2}(t^2+2t+4) = (-2)^2+2(-2)+4$$

$$= 4-4+4 = 4$$

14.

$$\lim_{x\to 3}\frac{2x^3-6x^2+x-3}{x-3} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 3}\frac{2x^2(x-3)+1(x-3)}{x-3}$$

$$= \lim_{x\to 3}\frac{(x-3)(2x^2+1)}{x-3}$$

$$= \lim_{x\to 3}(2x^2+1) = 2(3)^2+1 = 18+1 = 19$$

15.

$$\lim_{r\to 2}\frac{r^2+4r+3}{r^2-9} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \frac{(2)^2+4(2)+3}{(2)^2-9} = \frac{4+8+11}{4-9} = -\frac{23}{5}$$

16.

$$\lim_{x\to 2}\frac{x^2-7x+10}{x-2} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 2}\frac{x^2-5x-2x+10}{x-2}$$

$$= \lim_{x\to 2}\frac{x(x-5)-2(x-5)}{x-2}$$

$$= \lim_{x\to 2}\frac{(x-2)(x-5)}{x-2}$$

$$= \lim_{x\to 2}(x-5) = 2-5 = -3$$

17.

$$\lim_{y\to -2}\frac{-2y-4}{y+2} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{y\to -2}\frac{-2(y+2)}{y+2} = \lim_{y\to -2}(-2) = -2$$

18.

$$\lim_{x\to -1}\frac{x^2+3x+2}{x^2-x-2} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to -1}\frac{x^2+2x+x+2}{x^2-2x+x-2} = \lim_{x\to -1}\frac{x(x+2)+1}{x(x-2)+1}$$

$$= \lim_{x\to -1}\frac{(x+1)(x+2)}{(x+1)(x-2)} = \lim_{x\to -1}\frac{x+2}{x-2}$$

$$= \frac{-1+2}{-1-2} = -\frac{1}{3}$$

19.

$$\lim_{x\to 3}\frac{2x^2+7x-15}{x-3} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \frac{2(3)^2+7(3)-15}{3-3} = \frac{18+21-15}{0}$$

$$= \frac{24}{0} = +\infty$$

20.

$$\lim_{t\to -1}\frac{t^2+3t+2}{t^2-t-2} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{t\to -1}\frac{t^2+2t+t+2}{t^2-2t+t-2}$$

$$= \lim_{t\to -1}\frac{t(t+2)+1(t+2)}{t(t-2)+1(t-2)}$$

$$= \lim_{t\to -1}\frac{(t+2)(t+1)}{(t+1)(t-2)} = \lim_{t\to -1}\frac{t+2}{t-2}$$

$$= \frac{\lim_{t\to 1}(t+2)}{\lim_{t\to 1}(t-2)} = \frac{-1+2}{-1-2} = -\frac{1}{3}$$

In Exercise (21-26), find the limits

21. $\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1}$, $\left(\frac{0}{0}\right)$ form

$$= \lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \times \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$= \lim_{x\to -1} \frac{\left(\sqrt{x^2+8}\right)^2-(3)^2}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{(x+1)(x-1)}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim_{x\to -1} \frac{(x-1)}{\sqrt{x^2+8}+3}$$

$$= \lim_{x\to -1} \frac{-1-1}{\sqrt{(-1)^2+8}+3} = \frac{-2}{\sqrt{9}+3} = \frac{-2}{3+3}$$

$$= \frac{-2}{3+3} = \frac{-2}{6} = -\frac{1}{3}$$

22.

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} \times \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{\left(\sqrt{x+3}\right)^2-(2)^2}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{x+3-4}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{x-1}$$

$$= \lim_{x\to 1} \left(\sqrt{x+3}+2\right) = \sqrt{1+3}+2$$

$$= \sqrt{4}+2 = 2+2 = 4$$

23.

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \times \frac{\sqrt{x}+2}{\sqrt{x}+2}$$

$$= \lim_{x\to 4} \frac{\left(\sqrt{x}\right)^2-(2)^2}{(x-4)\left(\sqrt{x}+2\right)}$$

$$= \lim_{x\to 4} \frac{x-4}{(x-4)\left(\sqrt{x}+2\right)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2}$$

$$= \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$$

24.

$$\lim_{x\to 1} \frac{x^3-1}{\sqrt{x}-1} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 1} \frac{(x)^3-(1)^3}{\sqrt{x}-1} \times \frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(x^2+x+1\right)\left(\sqrt{x}+1\right)}{\left(\sqrt{x}\right)^2-(1)^2}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x}+1\right)\left(x^2+x+1\right)}{x-1}$$

$$= \lim_{x\to 1} \left(\sqrt{x}+1\right)\left(x^2+x+1\right)$$

$$= \left(\sqrt{1}+1\right)\left(1^2+1+1\right) = (1+1)(1+1+1)$$

$$= (2)(3) = 6$$

25.

$$\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{h\to 0} \frac{(x+h-x)(x+h+x)}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x+h) = 2x+0 = 2x$$

26.

$$\lim_{x\to 9} \frac{9x-x^2}{3-\sqrt{x}} \quad , \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 9} \frac{x(9-x)}{3-\sqrt{x}}$$

$$= \lim_{x\to 9} \frac{x(9-x)}{3-\sqrt{x}} \times \frac{3+\sqrt{x}}{3+\sqrt{x}}$$

$$= \lim_{x\to 9} \frac{x(9-x)(3+\sqrt{x})}{(3)^2-(\sqrt{x})^2}$$

$$= \lim_{x\to 9} \frac{x(9-x)(3+\sqrt{x})}{9-x}$$

$$= \lim_{x\to 9} x(3+\sqrt{x}) = 9(3+\sqrt{9})$$

$$= 9(3+3) = 9(6) = 54$$

27.

$$\text{Let } f(x) = \begin{cases} x-1 & x \le 3 \\ 3x-7 & x > 3 \end{cases}$$

Solution

i) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x-1) = 3-1 = 2$

ii) $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (3x-7) = 3(3)-7 = 2$

iii) $\lim_{x\to 3} f(x)$

since $\lim_{x\to 3^-} f(x) = 2 = \lim_{x\to 3^+} f(x)$

Therefore $\lim_{x\to 3} f(x) = 3$

28.

$$\text{Let } f(x) = \begin{cases} x-2 & x < 0 \\ x^2 & 0 \le x \le 2 \\ 2x & x > 2 \end{cases}$$

Solution

i) $\lim_{x\to 0} f(x)$

L.H.L

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x-2) = 0-2 = -2$$

R.H.L

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = (0)^2 = 0$$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

Therefore $\lim_{x\to 0} f(x)$ *does not exist*

ii) $\lim_{x\to 2} f(x)$

L.H.L

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = (2)^2$$

R.H.L

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2x = 2(2)$$

Since $\lim_{x\to 2^-} f(x) = 4 = \lim_{x\to 2^+} f(x)$

Therefore $\lim_{x\to 2} f(x) = 4$

29.

$$\text{Let } f(x) = \begin{cases} \frac{x^2-4}{x+2} & x \neq -2 \\ C & x = -2 \end{cases}$$

Find c so that $f(-2) = \lim_{x\to -2} f$

Solution

Limit

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{x^2-4}{x+2}$$

$$= \lim_{x\to -2} \frac{(x+2)}{x}$$

$$= \lim_{x\to -2} (x-2)$$

Value

$$f(-2) = C$$

Since $f(-2) = \lim_{x\to -2} f(x)$

Therefore $C = -4$

In Exercise (30-33), find each exists

30.

$$f(x) = \frac{|x-4|}{x-4}, \quad a = 4$$

Solution

i) $\lim_{x\to a^-} f(x)$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{|x-4|}{x-4}$$

$$= \lim_{x\to 4^-} \frac{-(x-4)}{x-4} = -$$

ii) $\lim_{x\to a^+} f(x)$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{|x-4|}{x-4} = \lim_{x\to 4^+} \frac{x-4}{x-4} = 1$$

iii) $\lim_{x\to a} f(x)$

Since $\lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$

Therefore $\lim_{x\to 4} f(x)$ does not exist.

30.

$$f(x) = \frac{x+2}{|x+2|} \quad , \quad a = 2$$

Solution

i) $\lim_{x\to a^-} f(x)$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x+2}{|x+2|} = \lim_{x\to 2^-} \frac{x+2}{x+2} = 1$$

ii) $\lim_{x\to a} f(x)$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x+2}{|x+2|}$$

$$= \lim_{x\to 2^+} \frac{x+2}{x+2} = 1$$

iii) $\lim_{x\to a} f(x)$

Since $\lim_{x\to 2^-} f(x) = 1 = \lim_{x\to 2^+} f(x)$

Therefore $\lim_{x\to 2} f(x) = 1$.

31.

$$f(x) = \frac{|x+3|}{x+3} \qquad , a = -3$$

Solution

i) $\lim_{x\to a^-} f(x)$

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \frac{|x+3|}{x+3}$$

$$= \lim_{x\to -3^-} \frac{-(x+3)}{x+3} = -1$$

ii) $\lim_{x\to a^+} f(x)$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{|x+3|}{x+3}$$

$$= \lim_{x\to -3^+} \frac{-(x+3)}{x+3} = -1$$

iii) $\lim_{x\to a} f(x)$

Since $\lim_{x\to -3^-} f(x) \neq \lim_{x\to -3^+} f(x)$

Therefore $\lim_{x\to -3} f(x)$ does not exist.

32.

$$f(x) = \frac{x-1}{|x-1|} \qquad , a = -1$$

Solution

i) $\lim_{x\to a^-} f(x)$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x-1}{|x-1|}$$

$$= \lim_{x\to -1^-} \frac{x-1}{-(x-1)} = -1$$

ii) $\lim_{x\to a^+} f(x)$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x-1}{|x-1|}$$

$$= \lim_{x\to -1^+} \frac{x-1}{-(x-1)} = -1$$

iii) $\lim_{x\to a} f(x)$

Since $\lim_{x\to -1^-} f(x) = -1 = \lim_{x\to -1^+} f(x)$

Therefore $\lim_{x\to -1} f(x) = -1$

Exercises 3.2

In Exercise (1-16), find the limits:

1. $\lim_{x\to+\infty}\left(2x^3-10x+7\right)$

$$=\lim_{x\to+\infty}2x^3=2(+\infty)^3=+\infty$$

2.

$$\lim_{x\to+\infty}\left(2+3x-3x^5\right)$$

$$=\lim_{x\to+\infty}\left(-3x^5\right)=-3(+\infty)^5=-\infty$$

3. $\lim_{x\to+\infty}\sqrt{x}=\sqrt{+\infty}=+\infty$

4. $\lim_{x\to-\infty}\sqrt{2-x}=\sqrt{2+\infty}=+\infty$

5.

$$\lim_{x\to+\infty}\frac{2x+1}{3x-5}$$

Dividing up and down by x

$$\lim_{x\to+\infty}\frac{2+\frac{1}{x}}{3-\frac{5}{x}}=\frac{\lim_{x\to+\infty}\left(2+\frac{1}{x}\right)}{\lim_{x\to+\infty}\left(3-\frac{5}{x}\right)}$$

$$=\frac{\lim_{x\to+\infty}2+\lim_{x\to+\infty}\frac{1}{x}}{\lim_{x\to+\infty}3-\lim_{x\to+\infty}\frac{5}{x}}=\frac{2+0}{3-0}=\frac{2}{3}$$

6.

$$\lim_{x\to+\infty}\frac{5x^2-4x}{2x^2+3}$$

Dividing up and down by x^2

$$=\lim_{x\to+\infty}\frac{5-\frac{4}{x}}{2+\frac{3}{x^2}}=\frac{\lim_{x\to+\infty}\left(5-\frac{4}{x}\right)}{\lim_{x\to+\infty}\left(2+\frac{3}{x^2}\right)}$$

$$=\frac{\lim_{x\to+\infty}5-\lim_{x\to+\infty}\frac{4}{x}}{\lim_{x\to+\infty}2+\lim_{x\to+\infty}\frac{3}{x^2}}=\frac{5-0}{2+0}=\frac{5}{2}$$

7. $\lim_{t\to-\infty}\frac{3}{t+2}$

$$=\frac{\lim_{t\to-\infty}3}{\lim_{t\to-\infty}(t+2)}=\frac{\lim_{t\to-\infty}}{\lim_{t\to-\infty}t+\lim_{t\to-\infty}2}$$

$$=\frac{3}{-\infty+2}=\frac{3}{-\infty}=0$$

8.

$$\lim_{x\to+\infty}\frac{1}{x-7}$$

$$=\frac{\lim_{x\to+\infty}1}{\lim_{x\to+\infty}(x-7)}=\frac{\lim_{x\to+\infty}1}{\lim_{x\to+\infty}x-\lim_{x\to+\infty}7}$$

$$=\frac{1}{+\infty-7}=\frac{1}{\infty}=0$$

9.

$$\lim_{x\to-\infty}\frac{x-2}{x^2+2x+3}$$

Dividing up and down by x^2

$$=\lim_{x\to-\infty}\frac{\frac{1}{x}-\frac{2}{x^2}}{1+\frac{2}{x}+\frac{3}{x}}=\frac{\lim_{x\to-\infty}\left(\frac{1}{x}-\frac{2}{x^2}\right)}{\lim_{x\to-\infty}\left(1+\frac{2}{x}+\frac{3}{x}\right)}$$

$$=\frac{\lim_{x\to-\infty}\frac{1}{x}-\lim_{x\to-\infty}\frac{2}{x^2}}{\lim_{x\to-\infty}1+\lim_{x\to-\infty}\frac{2}{x}+\lim_{x\to-\infty}\frac{3}{x}}=\frac{0-0}{1+0+0}=0$$

10.

$$\lim_{y\to+\infty}\frac{5y^2+4}{3y^2-y}$$

Dividing up and down by y^2

$$=\lim_{y\to+\infty}\frac{5+\frac{4}{y^2}}{3-\frac{1}{y}}=\frac{\lim_{y\to+\infty}\left(5+\frac{4}{y^2}\right)}{\lim_{y\to+\infty}\left(3-\frac{1}{y}\right)}$$

$$= \frac{\lim_{y\to+\infty} 5 + \lim_{y\to+\infty} \frac{4}{y^2}}{\lim_{y\to+\infty} 3 - \lim_{y\to+\infty} \frac{1}{y}} = \frac{5+0}{3-0} = \frac{5}{3}$$

11. $\lim_{x\to+\infty} \frac{7-6x^5}{x+3}$

Dividing up and down by x

$$= \lim_{x\to+\infty} \frac{\frac{7}{x} - 6x^4}{1+\frac{3}{x}} = \frac{\lim_{x\to+\infty}\left(\frac{7}{x} - 6x^4\right)}{\lim_{x\to+\infty}\left(1+\frac{3}{x}\right)}$$

$$= \frac{\lim_{x\to+\infty} \frac{7}{x} - \lim_{x\to+\infty} 6x^4}{\lim_{x\to+\infty} 1 + \lim_{x\to+\infty} \frac{3}{x}} = \frac{0-(+\infty)}{1+0} = -\infty$$

12.

$$\lim_{x\to-\infty} \frac{5-2x^3}{x^2+1}$$

Dividing up and down by x^2

$$= \lim_{x\to-\infty} \frac{\frac{5}{x^2} - 2x}{1+\frac{1}{x^2}} = \frac{\lim_{x\to-\infty}\left(\frac{5}{x^2} - 2x\right)}{\lim_{x\to-\infty}\left(1+\frac{1}{x^2}\right)}$$

$$= \frac{\lim_{x\to-\infty} \frac{5}{x^2} - \lim_{x\to-\infty} 2x}{\lim_{x\to-\infty} 1 + \lim_{x\to-\infty} \frac{1}{x^2}} = \frac{0-2(-\infty)}{1+0} = +\infty$$

13. $\lim_{t\to+\infty} \frac{6-t^3}{7t^3+1}$

Dividing up and down by t^3

$$= \lim_{t\to+\infty} \frac{\frac{6}{t^3} - 1}{7+\frac{1}{t^3}} = \frac{\lim_{t\to+\infty}\left(\frac{6}{t^3} - 1\right)}{\lim_{t\to+\infty}\left(7+\frac{1}{t^3}\right)}$$

$$= \frac{\lim_{t\to+\infty} \frac{6}{t^3} - \lim_{t\to+\infty} 1}{\lim_{t\to+\infty} 7 + \lim_{t\to+\infty} \frac{1}{t^3}} = \frac{0-1}{7+0} = -\frac{1}{7}$$

14.

$$\lim_{x\to+\infty} \frac{3x^3 - x + 1}{6x^3 + 2x^2 - 7}$$

Dividing up and down by x^3

$$= \lim_{x\to+\infty} \frac{3 - \frac{1}{x^2} + \frac{1}{x^3}}{6 + \frac{2}{x} - \frac{7}{x^3}} = \frac{\lim_{x\to+\infty}\left(3 - \frac{1}{x^2} + \frac{1}{x^3}\right)}{\lim_{x\to+\infty}\left(6 + \frac{2}{x} - \frac{7}{x^3}\right)}$$

$$= \frac{\lim_{x\to+\infty} 3 - \lim_{x\to+\infty} \frac{1}{x^2} + \lim_{x\to+\infty} \frac{1}{x^3}}{\lim_{x\to+\infty} 6 + \lim_{x\to+\infty} \frac{2}{x} - \lim_{x\to+\infty} \frac{7}{x^3}} = \frac{3-0+0}{6+0-0}$$

$$= \frac{3}{6} = \frac{1}{2}$$

15.

$$\lim_{x\to-\infty} \frac{x^2+2}{x-1}$$

Dividing up and down by x

$$= \lim_{x\to-\infty} \frac{x + \frac{2}{x}}{1 - \frac{1}{x}} = \frac{\lim_{x\to-\infty}\left(x + \frac{2}{x}\right)}{\lim_{x\to-\infty}\left(1 - \frac{1}{x}\right)}$$

$$= \frac{\lim_{x\to-\infty} x + \lim_{x\to-\infty} \frac{2}{x}}{\lim_{x\to-\infty} 1 - \lim_{x\to-\infty} \frac{1}{x}} = \frac{-\infty+0}{1-0} = -\infty$$

16.

$$\lim_{x\to-\infty} \frac{4x-3}{x^2+1}$$

Dividing up and down by x^2

$$= \lim_{x\to-\infty} \frac{\frac{4}{x} - \frac{3}{x^2}}{1+\frac{1}{x^2}} = \frac{\lim_{x\to-\infty}\left(\frac{4}{x} - \frac{3}{x^2}\right)}{\lim_{x\to-\infty}\left(1+\frac{1}{x^2}\right)}$$

$$= \frac{\lim_{x\to-\infty} \frac{4}{x} - \lim_{x\to-\infty} \frac{3}{x^2}}{\lim_{x\to-\infty} 1 + \lim_{x\to-\infty} \frac{1}{x^2}} = \frac{0-0}{1+0} = 0$$

In Exercise (17-22) ,for the given f(x) ,express each of the following limits as $+\infty$, $-\infty$ or DNE (Does not exist):

17. $f(x)=\dfrac{1}{x-2}$, $a=2$

i) $\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}\dfrac{1}{x-2}=-\infty$

ii) $\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}\dfrac{1}{x-2}=+\infty$

iii)

Since $\lim\limits_{x\to 2^-} f(x)\neq \lim\limits_{x\to 2^+} f(x)$

Therefore $\lim\limits_{x\to 2} f(x)$ DNE.

18. $f(x)=\dfrac{5}{4-x}$, $a=4$

i) $\lim\limits_{x\to 4^-} f(x)=\lim\limits_{x\to 4^-}\dfrac{5}{4-x}=+\infty$

ii) $\lim\limits_{x\to 4^+} f(x)=\lim\limits_{x\to 4^+}\dfrac{5}{4-x}=-\infty$

iii)

Since $\lim\limits_{x\to 4^-} f(x)\neq \lim\limits_{x\to 4^+} f(x)$

Therefore $\lim\limits_{x\to 4} f(x)$ DNE.

19. $f(x)=\dfrac{-4}{2x+1}$, $a=-\dfrac{1}{2}$

i) $\lim\limits_{x\to \frac{1}{2}^-} f(x)=\lim\limits_{x\to \frac{1}{2}^-}\dfrac{4}{2x+1}=+\infty$

ii) $\lim\limits_{x\to \frac{1}{2}^+} f(x)=\lim\limits_{x\to \frac{1}{2}^+}\dfrac{-4}{2x+1}=-\infty$

iii)

Since $\lim\limits_{x\to \frac{1}{2}^-} f(x)\neq \lim\limits_{x\to \frac{1}{2}^+} f(x)$

Therefore $\lim\limits_{x\to \frac{1}{2}} f(x)$ DNE.

20. $f(x)=\dfrac{2x^2}{(x+8)^2}$, $a=-8$

i) $\lim\limits_{x\to -8^-} f(x)=\lim\limits_{x\to -8^-}\dfrac{2x^2}{(x+8)^2}$

$=\dfrac{2(-8)^2}{(-8+8)^2}=\dfrac{128}{0}=+\infty$

ii)

$\lim\limits_{x\to -8^+} f(x)=\lim\limits_{x\to -8^+}\dfrac{2x^2}{(x+8)^2}$

$=\dfrac{2(-8)^2}{(-8+8)^2}=\dfrac{128}{0}=+\infty$

iii)

Since $\lim\limits_{x\to -8^-} f(x)=+\infty=\lim\limits_{x\to -8^+} f($

Therefore $\lim\limits_{x\to -8} f(x)=+\infty$

21. $f(x)=\dfrac{3x^2}{x^2-x-2}$, $a=-1$

i)

$\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^-}\dfrac{3x^2}{x^2-x-2}$

$=\lim\limits_{x\to -1^-}\dfrac{3x^2}{x^2-2x+x-2}=\lim\limits_{x\to -1}\dfrac{}{(x+}$

$=\dfrac{3(-1)^2}{(-1+1)(-1-2)}=\dfrac{3}{-(0)(-3)}=\dfrac{3}{0}$

ii)

$\lim\limits_{x\to -1^+} f(x)=\lim\limits_{x\to -1^+}\dfrac{3x^2}{x^2-x-2}$

$=\lim\limits_{x\to -1^+}\dfrac{3x^2}{x^2-2x+x-2}=\lim\limits_{x\to -1^+}\dfrac{}{(x+1}$

$=\dfrac{3(-1)^2}{(-1+1)(-1-2)}=\dfrac{3}{+(0)(-3)}=-\dfrac{3}{0}$

iii)

Since $\lim\limits_{x\to -1^-} f(x)\neq \lim\limits_{x\to -1^+} f(x)$

Therefore $\lim\limits_{x\to -1} f(x)$ *DNE*

22. $f(x)=\dfrac{4x}{x^2-4x+3}$, $a=1$

i)

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{4x}{x^2-4x+3}$$

$$= \lim_{x\to 1} \frac{4x}{x^2-3x-x+3} = \lim_{x\to 1} \frac{4x}{(x-1)(x-3)}$$

$$= \frac{4(1)}{(1-1)(1-3)} = \frac{4}{(0)(-2)} = +\frac{4}{0} = +\infty$$

ii)

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{4x}{x^2-4x+3}$$

$$= \lim_{x\to 1^+} \frac{4x}{x^2-3x-x+3} = \lim_{x\to 1^+} \frac{4x}{(x-1)(x-3)}$$

$$= \frac{4(1)}{(1-1)(1-3)} = \frac{4}{+(0)(-2)} = -\frac{4}{0} = -\infty$$

iii)

Since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

Therefore $\lim_{x\to 1} f(x)$ *DNE* .

In Exercise (23-28), express each limit in terms of e.

23. $\lim_{x\to+\infty}\left(1+\frac{3}{x}\right)^{2x}$

$$= \lim_{x\to+\infty}\left(1+\frac{3}{x}\right)^{2x\cdot\frac{3}{3}} = \lim_{x\to+\infty}\left(1+\frac{3}{x}\right)^{6\cdot\frac{x}{3}}$$

$$= \left[\lim_{x\to+\infty}\left(1+\frac{3}{x}\right)^{\frac{x}{3}}\right]^6 = e^6$$

24. $\lim_{x\to 0}(1+2x)^{\frac{1}{x}}$

$$= \lim_{x\to 0}(1+2x)^{2\cdot\frac{1}{2x}}$$

$$= \left[\lim_{x\to 0}(1+2x)^{\frac{1}{2x}}\right]^2 = e^2$$

25.

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{\frac{x}{3}}$$

$$= \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{\frac{1}{3}x} = \left[\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{\frac{1}{3}}$$

$$= e^{\frac{1}{3}} = \sqrt[3]{e}$$

26.

$$\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^{x}$$

$$= \lim_{x\to+\infty}\left(1+\left(-\frac{1}{x}\right)\right)^{-x\cdot(-1)}$$

$$= \left[\lim_{x\to+\infty}\left(1+\left(-\frac{1}{x}\right)\right)^{x}\right]^{-1} = e^{-1} = \frac{1}{e}$$

27.

$$\lim_{x\to 0}(1+2x^2)^{\frac{1}{x^2}}$$

$$= \lim_{x\to 0}(1+2x^2)^{2\cdot\frac{1}{2x^2}}$$

$$= \left[\lim_{x\to 0}(1+2x^2)^{\frac{1}{2x^2}}\right]^2 = e^2$$

28.

$$\lim_{x\to+\infty}\left(\frac{x}{1+x}\right)^{x}$$

$$= \lim_{x\to+\infty}\left(\frac{1+x}{x}\right)^{-x} = \lim_{x\to+\infty}\left(\frac{1}{x}+\frac{x}{x}\right)^{-x}$$

$$= \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{-x} = \left[\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{-1}$$

$$= e^{-1} = \frac{1}{e}$$

$$= \lim_{\theta \to 0} \frac{(1)^2 - (\cos\theta)^2}{\sin^2\theta(1+\cos\theta)} = \lim_{\theta \to 0} \frac{1-\cos^2\theta}{\sin^2\theta(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\sin^2\theta}{\sin^2\theta(1+\cos\theta)} = \lim_{\theta \to 0} \frac{1}{1+\cos\theta}$$

$$= \frac{1}{1+\cos 0} = \frac{1}{1+1} = \frac{1}{2}$$

11.

$$\lim_{\theta \to 0} \frac{\sin^2\theta}{\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta.\sin\theta}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta}.\lim_{\theta \to 0}\sin\theta$$

$$= (1)\sin 0 = (1)(0) = 0$$

12.

$$\lim_{x \to 0} \frac{1-\cos 2x}{x^2}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} = 2\lim_{x \to 0}\left(\frac{\sin x}{x}\right)^2$$

$$= 2\left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2 = 2(1) = 2$$

13.

$$\lim_{\theta \to 0} \frac{\sin 5\theta}{\sin 4\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin 5\theta}{\sin 4\theta} \times \frac{5\theta}{5\theta} \times \frac{4\theta}{4\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin 5\theta}{5\theta} \cdot \frac{4\theta}{\sin 4\theta} \cdot \frac{5\theta}{4\theta}$$

$$= \frac{5}{4}\lim_{\theta \to 0} \frac{\sin 5\theta}{5\theta}.\lim_{\theta \to 0} \frac{4\theta}{\sin 4\theta} = \frac{5}{4}(1)(1) = \frac{5}{4}$$

14.

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\sin 2\theta}$$

$$= \lim_{\theta \to 0} \frac{1-\cos\theta}{\sin 2\theta} \times \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta \to 0} \frac{(1)^2 - (\cos\theta)^2}{\sin 2\theta(1+\cos\theta)}$$

Using formula $(\sin 2\theta = 2\sin\theta\cos\theta)$

$$= \lim_{\theta \to 0} \frac{1-\cos^2\theta}{2\sin\theta\cos\theta(1+\cos\theta)}$$

Using formula $(\sin^2\theta = 1-\cos^2\theta)$

$$= \lim_{\theta \to 0} \frac{\sin^2\theta}{2\sin\theta\cos\theta(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{2\cos\theta(1+\cos\theta)}$$

$$= \frac{\sin 0}{2\cos 0(1+\cos 0)} = \frac{0}{2(1)(1+1)} = 0$$

15.

$$\lim_{t \to 0} \frac{\sin(\sin t)}{\sin t}$$

Let $\sin t = \theta$

As $t \to 0, \theta \to 0$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

16.

$$\lim_{x \to 0} \frac{x^2 - x + \sin x}{2x}$$

$$= \lim_{x \to 0} \frac{x(x-1)+\sin x}{2x}$$

$$= \lim_{x \to 0} \frac{x(x-1)}{2x} + \lim_{x \to 0} \frac{\sin x}{2x}$$

$$= \lim_{x \to 0} \frac{x-1}{2} + \frac{1}{2}\lim_{x \to 0} \frac{\sin x}{x}$$

$$= \frac{0-1}{2} + \frac{1}{2}(1) = -\frac{1}{2} + \frac{1}{2} = 0$$

Exercises 3.4

In Exercise (1-10), determine whether there are any discontinuities and if so, where they occur.

1. $f(x)=2x^2+3x+4$

We see that the function $f(x)=2x^2+3x+4$ is continuous for any real x because f is defined for any real numbers and $\lim_{x\to a} f(x)$ exists for all $a\in R$

2. $f(x)=\dfrac{1}{x+1}$

We see that f is not defined when

$$x+1=0$$

Since the left side of equation equals 0 when $x=-1$, so the function is discontinuous at this value..

3. $f(x)=\dfrac{x^4}{3}$

We see that the function $f(x)=\dfrac{x^4}{3}$ is continuous for any real x because f is defined for any real numbers and $\lim_{x\to a} f(x)$ exists for all $a\in R$.

4. $f(x)=\dfrac{2x^3}{x-2}$

We see that f is not defined when $x-2=0$. Since the left side of equation equals 0 when $x=2$, so the function is discontinuous at this value.

5. $f(x)=\dfrac{2}{9-3x}$

We see that f is not defined when

$9-3x=0\Rightarrow 3(3-x)=0\Rightarrow 3-x=0$.

Since the left side of equation equals 0 when $x=3$, so the function is discontinuous at this value.

6. $f(x)=\dfrac{x-2}{x^2-4x}$

We see that f is not defined when $x^2-4x=0\Rightarrow x(x-4)$. Since the left sid of equation equals 0 when $x=0,4$, so the function is discontinuous at these values.

7. $f(x)=\dfrac{10}{x^2-3x-10}$

We see that f is not defined when

$x^2-3x-10=0.$

$x^2-5x+2x-10=0$

$x(x-5)+2(x-5)=0$

$(x-5)(x+2)=0$

Since the left side of equation equals 0 when $x=-2,5$, so the function is discontinuous at these values.

8. $f(x)=\dfrac{5}{(x-3)(x^2-16)}$

We see that f is not defined when

$$(x-3)(x^2-16)=0$$

$$(x-3)(x-4)(x+4)$$

Since the left side of equation equals 0 when $x=3,4,-4$, so the function is discontinuous at these values.

9. $f(x)=\dfrac{20}{(5-x)(4-x^2)}$

We see that f is not defined when

$$(5-x)(4-x^2)=0$$

$$(5-x)(2-x)(2+x)=0$$

Since the left side of equation equals 0 when $x = 5, 2, -2$, so the function is discontinuous at these values.

10. $f(x) = \dfrac{4x-2}{x^3 - x^2 - 6x}$

We see that f is not defined when

$$x^3 - x^2 - 6x = 0$$
$$x\left[x^2 - x - 6\right] = 0$$
$$x\left[x^2 - 3x + 2x - 6\right] = 0$$
$$x\left[x(x-3) + 2(x-3)\right] = 0$$
$$x(x-3)(x+2) = 0$$

Since the left side of equation equals 0 when $x = 0, 3, -2$, so the function is discontinuous at these values.

In Exercises (11-16), Classify the Discontinuities of f as removable, Jump, or infinity.

Solutions:

11.

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases}$$

L.H.L

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left(x^2 - 1\right) = 1^2 - 1 = 0$$

R.H.L

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4 - x) = 4 - 1 = 3$$

Value

$$f(x) = 4 - x$$
$$f(1) = 4 - 1 = 3$$

Since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$ *but* $f(1)$ *defined*

Therefore it is Jump discontinuity at $x = 1$.

12.

$$f(x) = \begin{cases} x^3 & \text{if } x \leq 1 \\ 3 - x & \text{if } x > 1 \end{cases}$$

L.H.L

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^3 = (1)^3 = 1$$

R.H.L

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3 - x) = 3 - 1 = 2$$

Value

$$f(x) = x^3$$
$$f(1) = (1)^3 = 1$$

Since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$ *but* $f(1)$ *defined*

Therefore it is Jump discontinuity at $x = 1$.

13.

$$f(x) = \begin{cases} |x+3| & \text{if } x \neq -2 \\ 2 & \text{if } x = -2 \end{cases}$$

L.H.L

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} |x+3|$$
$$= \lim_{x \to -2^-} (x+3) = -2 + 3 = 1$$

R.H.L

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} |x+3|$$
$$= \lim_{x \to -2^+} (x+3) = -2 + 3 = 1$$

Value

$$f(x) = 2$$
$$f(-2) = 2$$

Since $\lim_{x \to -2^-} f(x) = 1 = \lim_{x \to -2^+} f(x)$

Therefore $\lim_{x \to -2} f(x) = 1$

Also $\lim_{x \to -2} f(x) = f(-2)$

So it is removable discontinuity at $x = -2$.

14.

$$f(x) = \begin{cases} |x-1| & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

L.H.L

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} |x-1|$$
$$= \lim_{x \to 1^-} -(x-1) = -(1-1) = 0$$

R.H.L

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} |x-1|$$
$$= \lim_{x\to 1^+} (x-1) = 1-1 = 0$$

Value

$$f(x) = 1$$
$$f(1) = 1$$

Since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

Therefore it is removable discontinuity at $x = 1$.

15.

$$f(x) = \begin{cases} x^2+1 & \text{if } x<1 \\ 1 & \text{if } x=1 \\ x+1 & \text{if } x>1 \end{cases}$$

L.H.L

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2+1) = 1^2+1 = 2$$

R.H.L

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+1) = 1+1 = 2$$

Since $\lim_{x\to 1^-} f(x) = 2 = \lim_{x\to 1^+} f(x)$

Therefore $\lim_{x\to 1} f(x) = 2$

Value

$$f(x) = 1$$
$$f(1) = 1$$

Since $\lim_{x\to 1} f(x) = f(1)$, *therefore it is removable discontinuity at* $x = 0$.

16.

$$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x<0 \\ \dfrac{4}{x} & \text{if } x>0 \end{cases}$$

L.H.L

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x^2} = +\infty$$

R.H.L

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{4}{x} = +\infty$$

Since $\lim_{x\to 0^-} f(x) = +\infty = \lim_{x\to 0^+} f(x)$

Therefore, it is infinite discontinuity.

In Exercise (37-40), find all numbers at which f is continuous.

Solutions:

17.

$$f(x) = \frac{3x-5}{2x^2-x-3}$$

We see that f is not defined when

$$2x^2-x-3 = 0$$
$$2x^2-3x+2x-3 = 0$$
$$x(2x-3)+1(2x-3) = 0$$
$$(x+1)(2x-3) = 0$$

Since the left side of equation equals 0 when $x = -1, \frac{3}{2}$, so the function is continuous at these value.

Hence f is continuous for all real number except $x = -1, \frac{3}{2}$.

18.

$$f(x) = \sqrt{2x-3} + x^2$$

We see that f is defined when

$$2x-3 \geq 0 \Rightarrow x \geq \frac{3}{2}.$$

Hence f is defined in the interval

$$\left[\frac{3}{2}, +\infty\right)$$

19.

$$f(x) = \frac{x^2-9}{x-3}$$

We see that f is not defined when $x = 3$, so f is not continuous at $x = 3$.

Hence f is continuous for all real number except $x=3$.

20.

$$f(x)=\frac{3x-5}{x^4-27x}$$

We see that f is not defined when

$$x^4-27x=0$$

$$x(x^3-27)=0$$

$$x(x-3)=0, \quad \therefore (x^2+3x+9=0 \text{ gives imaginary value})$$

Since the left side of equation equals 0 when $x=0,3$, so the function is not continuous at these values.

Hence f is continuous for all real number except $x=0,3$.

In Exercise(21-24), discuss the continuity of f at $x=a$.

Solutions:

21.

$$f(x)=\begin{cases}2x+5 & \text{if } x\le 2\\ 4x+1 & \text{if } x>2\end{cases} \qquad a=2$$

L.H.L

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(2x+5)=2(2)+5=9$$

R.H.L

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(4x+1)=4(2)+1=9$$

Value

$$f(x)=2x+5$$

$$f(2)=2(2)+5$$

$$f(2)=9$$

Since $\lim_{x\to 2^-} f(x)=f(2)=\lim_{x\to 2^+} f(x)$

Therefore f is continuous.

22.

$$f(x)=\begin{cases}3x-1 & \text{if } x<1\\ 4 & \text{if } x=1\\ 2x & \text{if } x>1\end{cases} \qquad a=1$$

L.H.L

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(3x-1)=3(1)-1=2$$

R.H.L

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} 2x=2(1)=2$$

Value

$$f(x)=4$$

$$f(1)=4$$

Since $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)\neq f(1)$

Therefore f is not continuous at $x=1$.

23.

$$f(x)=\begin{cases}x-1 & \text{if } x<3\\ 2x+1 & \text{if } x\ge 3\end{cases} \qquad a=3$$

L.H.L

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(x-1)=3-1=2$$

R.H.L

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(2x+1)=2(3)+1=7$$

Since $\lim_{x\to 3^-} f(x)\neq \lim_{x\to 3^+} f(x)$

Therefore f is not continuous at $x=3$.

24.

$$f(x)=\begin{cases}4x^2-8 & \text{if } x<2\\ 4x & \text{if } x=2\\ 2x^2 & \text{if } x>2\end{cases} \qquad a=2$$

L.H.L

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(4x^2-8)=4(2)^2-8$$

$$=16-8=8$$

R.H.L

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+} 2x^2=2(2)^2=8$$

Value

$$f(x)=4x$$

$$f(2)=4(2)=8$$

Since $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)$

Therefore f is continuous at $x=2$.

25.

Suppose that

$$f(x)=\begin{cases} c^2x & \text{if } x<1 \\ 3cx-2 & \text{if } x\geq 1 \end{cases}$$

Determine all values of c, such that f is continuous on R.

Solution:

L.H.L

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} c^2x=c^2(1)=c^2$$

R.H.L

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(3cx-2)=3c(1)-2=3c-2$$

Since f is continuous, therefore

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)$$

$$c^2=3c-2$$

$$c^2-3c+2=0$$

$$c^2-2c-c+2=0$$

$$c(c-2)-1(c-2)=0$$

$$(c-1)(c-2)=0$$

$$c=1,2$$

26.

Suppose that

$$f(x)=\begin{cases} 4x & x\leq -1 \\ cx+d & -1<x<2 \\ -5x & x\geq 2 \end{cases}$$

Solution:

L.H.L

$$\lim_{x\to -1^-} f(x)=\lim_{x\to -1^-} 4x=4(-1)=-4$$

R.H.L

$$\lim_{x\to -1^+} f(x)=\lim_{x\to -1^+}(cx+d)$$

$$=c(-1)+d=-c+d$$

Since f is continuous, therefore

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)$$

$$-4=-c+d$$

$$\Rightarrow c-d=4 \text{ ------- } (1)$$

Also

L.H.L

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(cx+d)=c(2)+d=2c+d$$

R.H.L

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(-5x)=-5(2)=-10$$

Since f is continuous, therefore

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)$$

$$2c+d=-10 \text{ _____ } (2)$$

Adding (1) *and* (2)

$$c-d=4$$

$$2c+d=-10$$

$$3c=-6$$

$$\Rightarrow c=-2$$

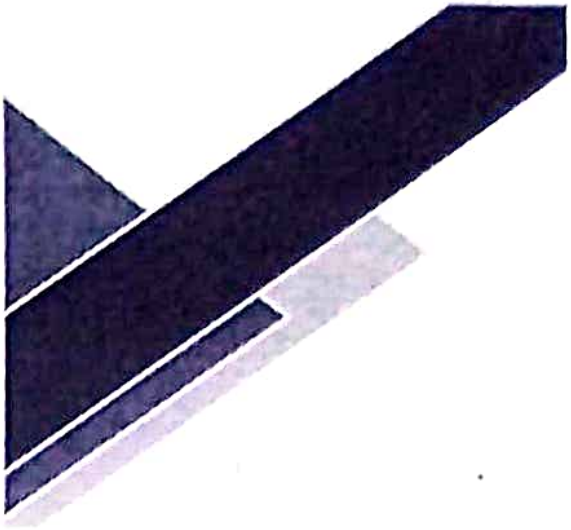
Put $c=-2$ *in* (1)

$$-2-d=4$$

$$-d=4+2\Rightarrow d=-6$$

CHAPTER 4

THE DERIVATIVE



Exercises 4.1

n exercise $(1-8)$, for each of the following functions, determine the difference quotient nd average rate of change in the value of y vhile moving from $x=1$ to $x=3$.

Solutions:

. $y=f(x)=2x^2$

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\frac{2(x+\Delta x)^2-2x^2}{\Delta x}$$

$$=\frac{2\left(x^2+2x\Delta x+\Delta x^2\right)-2x^2}{\Delta x}$$

$$=\frac{4x\Delta x+2\Delta x^2}{\Delta x}$$

$$=\frac{\Delta x\,(4x+2\Delta x)}{\Delta x}$$

$$=4x+2\Delta x$$

Since $x=1$, $x+\Delta x=3 \Rightarrow \Delta x=2$.

Now,

$$\frac{\Delta y}{\Delta x}=4(1)+2(2)=4+4=8$$

2. $y=f(x)=5x^2$

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\frac{5(x+\Delta x)^2-5x^2}{\Delta x}$$

$$=\frac{5\left(x^2+2x\Delta x+\Delta x^2\right)-5x^2}{\Delta x}$$

$$=\frac{10x\Delta x+5\Delta x^2}{\Delta x}$$

$$=\frac{\Delta x\,(10x+5\Delta x)}{\Delta x}$$

$$=10x+5\Delta x$$

Since $x=1$, $x+\Delta x=3 \Rightarrow \Delta x=2$.

Now,

$$\frac{\Delta y}{\Delta x}=10(1)+5(2)=10+10=20$$

3. $y=f(x)=x^2-2x+3$

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \frac{(x+\Delta x)^2 - 2(x+\Delta x) + 3 - (x^2 - 2x + 3)}{\Delta x}$$

$$= \frac{x^2 + 2x\Delta x + \Delta x^2 - 2x - 2\Delta x + 3 - x^2 + 2x - 3}{\Delta x}$$

$$= \frac{2x\Delta x + \Delta x^2 - 2\Delta x}{\Delta x}$$

$$= \frac{\Delta x(2x + \Delta x - 2)}{\Delta x}$$

$$= 2x + \Delta x - 2$$

Since $x = 1$, $x + \Delta x = 3 \Rightarrow \Delta x = 2$.

Now,

$$\frac{\Delta y}{\Delta x} = 2(1) + 2 - 2 = 2$$

4. $y = f(x) = x^3 - 2x^2 + 2$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{(x+\Delta x)^3 - 2(x+\Delta x)^2 + 2 - (x^2 - 2x^2 + 3)}{\Delta x}$$

$$= \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - 2(x^2 + 2x\Delta x + \Delta x^2) + 2 - x^3 + 2x^2 - 2}{\Delta x}$$

$$= \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - 2x^2 - 4x\Delta x - 2\Delta x^2 + 2 - x^3 + 2x^2 - 2}{\Delta x}$$

$$= \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - 4x\Delta x - 2\Delta x^2}{\Delta x}$$

$$= \frac{\Delta x(3x^2 + 3x\Delta x + \Delta x^2 - 4x - 2\Delta x)}{\Delta x}$$

$$= 3x^2 + 3x\Delta x + \Delta x^2 - 4x - 2\Delta x$$

Since $x = 1$, $x + \Delta x = 3 \Rightarrow \Delta x = 2$.

Now,

$$\frac{\Delta y}{\Delta x} = 3(1)^2 + 3(1)(2) + (2)^2 - 4(1) - 2(2) = 3 + 6 + 4 - 4 - 4 = 5$$

5. $y = f(x) = 4x^2 - 2x$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{4(x+\Delta x)^2 - 2(x+\Delta x) - (4x^2 - 2x)}{\Delta x}$$

$$= \frac{4(x^2 + 2x\Delta x + \Delta x^2) - 2(x+\Delta x) - 4x^2 + 2x}{\Delta x}$$

$$= \frac{4x^2 + 8x\Delta x + 4\Delta x^2 - 2x - 2\Delta x - 4x^2 + 2x}{\Delta x}$$

$$= \frac{8x\Delta x + 4\Delta x^2 - 2\Delta x}{\Delta x} = \frac{\Delta x\,(8x + 4\Delta x - 2)}{\Delta x}$$

$$= 8x + 4\Delta x - 2$$

Since $x = 1$, $x + \Delta x = 3 \Rightarrow \Delta x = 2$

Now,

$$\frac{\Delta y}{\Delta x} = 8(1) + 4(2) - 2 = 14$$

6. $y = f(x) = 2x^2 - 8x + 10$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \frac{2(x + \Delta x)^2 - 8(x + \Delta x) + 10 - (2x^2 - 8x + 10)}{\Delta x}$$

$$= \frac{2(x^2 + 2x\Delta x + \Delta x^2) - 8(x + \Delta x) + 10 - 2x^2 + 8x - 10}{\Delta x}$$

$$= \frac{2x^2 + 4x\Delta x + 2\Delta x^2 - 8x - 8\Delta x + 10 - 2x^2 + 8x - 10}{\Delta x}$$

$$= \frac{4x\Delta x + 2\Delta x^2 - 8\Delta x}{\Delta x} = \frac{\Delta x\,(4x + 2\Delta x - 8)}{\Delta x}$$

$$= 4x + 2\Delta x - 8$$

Since $x = 1$, $x + \Delta x = 3 \Rightarrow \Delta x = 2$.

Now,

$$\frac{\Delta y}{\Delta x} = 4(1) + 2(2) - 8 = 4 + 4 - 8 = 0$$

7. $y = f(x) = -x^2 + 2x + 4$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \frac{-(x + \Delta x)^2 + 2(x + \Delta x) + 4 - (-x^2 + 2x + 4)}{\Delta x}$$

$$= \frac{-(x^2 + 2x\Delta x + \Delta x^2) + 2(x + \Delta x) + 4 - (-x^2 + 2x + 4)}{\Delta x}$$

$$= \frac{-x^2 - 2x\Delta x - \Delta x^2 + 2x + 2\Delta x + 4 + x^2 - 2x - 4}{\Delta x}$$

$$= \frac{-2x\Delta x - \Delta x^2 + 2\Delta x}{\Delta x} = \frac{\Delta x(-2x - \Delta x + 2)}{\Delta x}$$

$$= -2x - \Delta x + 2$$

Since $x = 1$, $x + \Delta x = 3 \Rightarrow \Delta x = 2$.

Now,

$$\frac{\Delta y}{\Delta x} = -2(1) - 2 + 2 = -2 - 2 + 2 = -2$$

8. $y = f(x) = 3x^2 + 4x - 5$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \frac{3(x + \Delta x)^2 + 4(x + \Delta x) - 5 - (3x^2 + 4x - 5)}{\Delta x}$$

$$= \frac{3(x^2 + 2x\Delta x + \Delta x^2) + 4(x + \Delta x) - 5 - (3x^2 + 4x - 5)}{\Delta x}$$

$$= \frac{3x^2 + 6x\Delta x + 3\Delta x^2 + 4x + 4\Delta x - 5 - 3x^2 - 4x + 5}{\Delta x}$$

$$= \frac{6x\Delta x + 3\Delta x^2 + 4\Delta x}{\Delta x} = \frac{\Delta x(6x + 3\Delta x + 4)}{\Delta x}$$

$$= 6x + 3\Delta x + 4$$

Since $x = 1$, $x + \Delta x = 3 \Rightarrow \Delta x = 2$.

Now,

$$\frac{\Delta y}{\Delta x} = 6(1) + 3(2) + 4 = 6 + 6 + 4 = 16$$

9. A ball is thrown straight up into the air. The height of the ball can be described as a function of time as

$$h(t) = -16t^2 + 128t$$

where $h(t)$ is height measured in feet and t is time measured in seconds.

a) Determine the average rate of change in the height between

i) $t = 0$ and $t = 2$ **ii)** $t = 0$ and $t = 4$

b) How long does it take for the ball to hit the ground?

Solution

$$h(t) = -16t^2 + 128t$$

a) $\dfrac{\Delta h}{\Delta t}=\dfrac{h(t+\Delta t)-h(t)}{\Delta t}$

$$=\frac{-16(t+\Delta t)^2+128(t+\Delta t)-\left(-16t^2+128t\right)}{\Delta t}$$

$$=\frac{-16\left(t^2+2t\Delta t+\Delta t^2\right)+128(t+\Delta t)-\left(-16t^2+128t\right)}{\Delta t}$$

$$=\frac{-16t^2-32t\Delta t-16\Delta t^2+128t+128\Delta t+16t^2-128t}{\Delta t}$$

$$=\frac{-32t\Delta t-16\Delta t^2+128\Delta t}{\Delta t}=\frac{\Delta t\left(-32t-16\Delta t+128\right)}{\Delta t}$$

$$\frac{\Delta h}{\Delta t}=-32t-16\Delta t+128$$

i) $t=0$ and $t=2$

Here $t=0$, $t+\Delta t=2 \Rightarrow \Delta t=2$

So, $\dfrac{\Delta h}{\Delta t}=-32(0)-16(2)+128=0-32+128=96$

ii) $t=0$ and $t=4$

Here $t=0$, $t+\Delta t=4 \Rightarrow \Delta t=4$

So, $\dfrac{\Delta h}{\Delta t}=-32(0)-16(4)+128=0-64+128=64$

b) When the ball hits the ground , $h(t)=0$.

$$-16t^2+128t=0$$

$$\Rightarrow -16t(t-8)=0$$

$$\Rightarrow t(t-8)=0 \Rightarrow t=0 \text{ and } t=8$$

After 8 seconds the ball hits the ground.

10. An object is dropped from a bridge which is 432 feet high. The height of the object can be determined as a function of time (since being dropped) according to the function

$$h(t)=432-12t^2$$

where $h(t)$ is height measured in feet and t is time measured in seconds.

a) Determine the average rate of change in height between

i) $t=0$ and $t=1$ **ii)** $t=0$ and $t=2$ **iii)** $t=0$ and $t=4$

b) How long does it take for the ball to hit the water?

Solution

$$h(t) = 432 - 12t^2$$

a) $$\frac{\Delta h}{\Delta t} = \frac{h(t+\Delta t) - h(t)}{\Delta t}$$

$$= \frac{432 - 12(t+\Delta t)^2 - (432 - 12t^2)}{\Delta t}$$

$$= \frac{432 - 12(t^2 + 2t\Delta t + \Delta t^2) - (432 - 12t^2)}{\Delta t}$$

$$= \frac{432 - 12t^2 - 24t\Delta t - 12\Delta t^2 - 432 + 12t^2}{\Delta t}$$

$$= \frac{-24t\Delta t - 12\Delta t^2}{\Delta t} = \frac{\Delta t(-24t - 12\Delta t)}{\Delta t}$$

$$\frac{\Delta h}{\Delta t} = -24t - 12\Delta t$$

i) $t = 0$ and $t = 1$

Here $t = 0$, $t + \Delta t = 1 \Rightarrow \Delta t = 1$

So, $\frac{\Delta h}{\Delta t} = -24(0) - 12(1) = -12$

ii) $t = 0$ and $t = 2$

Here $t = 0$, $t + \Delta t = 2 \Rightarrow \Delta t = 2$

So, $\frac{\Delta h}{\Delta t} = -24(0) - 12(2) = 0 - 24 = -24$

iii) $t = 0$ and $t = 4$

Here $t = 0$, $t + \Delta t = 4 \Rightarrow \Delta t = 4$

So, $\frac{\Delta h}{\Delta t} = -24(0) - 12(4) = 0 - 48 = -48$

b) When the ball hits the water , $h(t) = 0$.

$$-432 - 12t^2 = 0$$

$$\Rightarrow 12(36 - t^2) = 0$$

$$\Rightarrow 36 - t^2 = 0 \quad t^2 = 36 \Rightarrow t = 6$$

After 6 seconds the ball hits the water.

Exercises 4.2

In Exercise (1 – 12), using the limit approach,
a) Determine the derivative of f.
b) Determine the slope of tangent at $x = -1$ and $x = 2$.

Solutions:

1. $f(x) = 2x + 3$

a)

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{2(x+\Delta x)+3-(2x+3)}{\Delta x}$$

$$= \frac{2x+2\Delta x+3-2x-3}{\Delta x}$$

$$= \frac{2\Delta x}{\Delta x} = 2$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 2$$

$$\frac{dy}{dx} = 2 \quad \left(\because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \right)$$

b) $\left.\frac{dy}{dx}\right|_{x=-1} = 2$ and $\left.\frac{dy}{dx}\right|_{x=2} = 2$

2. $f(x) = -10$

a)

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{-10-(-10)}{\Delta x} = \frac{-10+10}{\Delta x} = 0$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 0$$

$$\frac{dy}{dx} = 0 \quad \left(\because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \right)$$

b) $\left.\frac{dy}{dx}\right|_{x=-1} = 0$ and $\left.\frac{dy}{dx}\right|_{x=2} = 0$

3. $f(x) = x^2 + 2$

a)

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{(x+\Delta x)^2 + 2 - (x^2+2)}{\Delta x}$$

$$= \frac{x^2 + 2x\Delta x + \Delta x^2 + 2 - x^2 - 2}{\Delta x}$$

$$= \frac{2x\Delta x + \Delta x^2}{\Delta x} = \frac{\Delta x(2x+\Delta x)}{\Delta x}$$

$$= 2x + \Delta x$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x)$$

$$\frac{dy}{dx} = 2x + 0 = 2x \quad \left(\because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \right)$$

b) $\left.\frac{dy}{dx}\right|_{x=-1} = 2(-1) = -2$

and $\left.\frac{dy}{dx}\right|_{x=2} = 2(2) = 4$

4. $f(x) = x^2 + 5$

a)

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{(x+\Delta x)^2 + 5 - (x^2+2)}{\Delta x}$$

$$= \frac{x^2 + 2x\Delta x + \Delta x^2 + 5 - x^2 - 2}{\Delta x}$$

$$= \frac{2x\Delta x + \Delta x^2}{\Delta x} = \frac{\Delta x(2x+\Delta x)}{\Delta x}$$

$$= 2x + \Delta x$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x)$$

$$\frac{dy}{dx} = 2x + 0 = 2x \quad \left(\because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \right)$$

b) $\left.\frac{dy}{dx}\right|_{x=-1} = 2(-1) = -2$

and $\left.\frac{dy}{dx}\right|_{x=2} = 2(2) = 4$

5. $f(x) = 3x^2 - 5x$

a)

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{3(x+\Delta x)^2 - 5(x+\Delta x) - (3x^2 - 5x)}{\Delta x}$$

$$= \frac{3(x^2 + 2x\Delta x + \Delta x^2) - 5(x+\Delta x) - (3x^2 - 5x)}{\Delta x}$$

$$= \frac{3x^2 + 6x\Delta x + 3\Delta x^2 - 5x - 5\Delta x - 3x^2 + 5x}{\Delta x}$$

$$= \frac{6x\Delta x + 3\Delta x^2 - 5\Delta x}{\Delta x}$$

$$= \frac{\Delta x(6x + 3\Delta x - 5)}{\Delta x}$$

$$= 6x + 3\Delta x - 5$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (6x + 3\Delta x - 5)$$

$$\frac{dy}{dx} = 6x + 3(0) - 5 = 6x - 5$$

b) $\left.\frac{dy}{dx}\right|_{x=-1} = 6(-1) - 5 = -6 - 5 = -11$

and $\left.\frac{dy}{dx}\right|_{x=2} = 6(2) - 5 = 12 - 5 = 7$

6. $f(x) = 10x^2 - 8x$

a)

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{10(x+\Delta x)^2 - 8(x+\Delta x) - (10x^2 - 8x)}{\Delta x}$$

$$= \frac{10(x^2 + 2x\Delta x + \Delta x^2) - 8(x+\Delta x) - (10x^2 - 8x)}{\Delta x}$$

$$= \frac{10x^2 + 20x\Delta x + 10\Delta x^2 - 8x - 8\Delta x - 10x^2 + 8x}{\Delta x}$$

$$= \frac{20x\Delta x + 10\Delta x^2 - 8\Delta x}{\Delta x}$$

$$= \frac{\Delta x(20x + 10\Delta x - 8)}{\Delta x} = 20x + 10\Delta x - 8$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (20x + 10\Delta x - 8)$$

$$\frac{dy}{dx} = 20x + 10(0) - 8 = 20x - 8 \qquad \left(\because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}\right)$$

b) $\left.\frac{dy}{dx}\right|_{x=-1} = 20(-1) - 8 = -20 - 8 = -28$

and $\left.\frac{dy}{dx}\right|_{x=2} = 20(2) - 8 = 40 - 8 = 32$

7. $f(x) = x^2 - 3x + 5$

a)

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{(x+\Delta x)^2 - 3(x+\Delta x) + 5 - (x^2 - 3x + 5)}{\Delta x}$$

$$= \frac{x^2 + 2x\Delta x + \Delta x^2 - 3x - 3\Delta x + 5 - x^2 + 3x - 5}{\Delta x}$$

$$= \frac{2x\Delta x + \Delta x^2 - 3\Delta x}{\Delta x}$$

$$= \frac{\Delta x(2x + \Delta x - 3)}{\Delta x} = 2x + \Delta x - 3$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 3)$$

$$\frac{dy}{dx} = 2x + 0 - 3 = 2x - 3 \qquad \left(\because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}\right)$$

b) $\left.\frac{dy}{dx}\right|_{x=-1} = 2(-1) - 3 \quad = -2 - 3 = -5$

and $\left.\frac{dy}{dx}\right|_{x=2} = 2(2)-3 = 4-3 = 1$

8. $f(x) = 5x^2 + 2x + 6$

a)

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \frac{5(x+\Delta x)^2 + 2(x+\Delta x) + 6 - (5x^2 + 2x + 6)}{\Delta x}$$

$$= \frac{5x^2 + 10x\Delta x + 5\Delta x^2 + 2x + 2\Delta x + 6 - 5x^2 - 2x - 6}{\Delta x}$$

$$= \frac{10x\Delta x + 5\Delta x^2 + 2\Delta x}{\Delta x}$$

$$= \frac{\Delta x(10x + 5\Delta x + 2)}{\Delta x} \quad = 10x + 5\Delta x + 2$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (10x + 5\Delta x + 2)$$

$$\frac{dy}{dx} = 10x + 5(0) + 2 = 10x + 2 \quad \left(\because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \right)$$

b) $\left.\frac{dy}{dx}\right|_{x=-1} = 10(-1) + 2 \quad = -10 + 2 = -8$

and $\left.\frac{dy}{dx}\right|_{x=2} = 10(2) + 2 \quad = 20 + 2 = 22$

9. $f(x) = -x^2 + 3x$

a)

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \frac{-(x+\Delta x)^2 + 3(x+\Delta x) - (-x^2 + 3x)}{\Delta x}$$

$$= \frac{-x^2 - 2x\Delta x - \Delta x^2 + 3x + 3\Delta x + x^2 - 3x}{\Delta x}$$

$$= \frac{-2x\Delta x - \Delta x^2 + 3\Delta x}{\Delta x}$$

$$= \frac{\Delta x(-2x - \Delta x + 3)}{\Delta x} = -2x - \Delta x + 3$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (-2x - \Delta x + 3)$$

$$\frac{dy}{dx} = -2x - 0 + 3 = -2x + 3 \qquad \left(\because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \right)$$

b) $\left.\frac{dy}{dx}\right|_{x=-1} = -2(-1) + 3 = 2 + 3 = 5$

and $\left.\frac{dy}{dx}\right|_{x=2} = -2(2) + 3 = -4 + 3 = -1$

10. $f(x) = -3x^2 - 10$

a)

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \frac{-3(x + \Delta x)^2 - 10 - (-3x^2 - 10)}{\Delta x}$$

$$= \frac{-3x^2 - 6x\Delta x - 3\Delta x^2 - 10 + 3x^2 + 10}{\Delta x}$$

$$= \frac{-6x\Delta x - 3\Delta x^2}{\Delta x} = \frac{\Delta x(-6x - 3\Delta x)}{\Delta x} = -6x - 3\Delta x$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (-6x - 3\Delta x)$$

$$\frac{dy}{dx} = -6x - 3(0) = -6x \qquad \left(\because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \right)$$

b) $\left.\frac{dy}{dx}\right|_{x=-1} = -6(-1) = 6$ and $\left.\frac{dy}{dx}\right|_{x=2} = -6(2) = -12$

11. $f(x) = \frac{2}{x}$

a)

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$=\frac{\frac{2}{(x+\Delta x)}-\frac{2}{x}}{\Delta x}$$

$$=\frac{2}{\Delta x}\left[\frac{1}{x+\Delta x}-\frac{1}{x}\right]$$

$$=\frac{2}{\Delta x}\left[\frac{x-x-\Delta x}{x(x+\Delta x)}\right]$$

$$=\frac{-2\Delta x}{x(x+\Delta x)}$$

$$=\frac{-2}{x(x+\Delta x)}$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{-2}{x(x+\Delta x)}\right)$$

$$\frac{dy}{dx}=\frac{-2}{x(x+0)}=-\frac{2}{x^2} \qquad \left(\because \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}\right)$$

b) $\left.\frac{dy}{dx}\right|_{x=-1}=-\frac{2}{(-1)^2}=-\frac{2}{1}=-2$

and $\left.\frac{dy}{dx}\right|_{x=2}=-\frac{2}{(2)^2}=-\frac{2}{4}=-\frac{1}{2}$

12. $f(x)=\frac{5}{x^2}$

a)

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\frac{\frac{5}{(x+\Delta x)^2}-\frac{5}{x^2}}{\Delta x}$$

$$=\frac{1}{\Delta x}\left[\frac{5}{(x+\Delta x)^2}-\frac{5}{x^2}\right]$$

$$=\frac{1}{\Delta x}\left[\frac{5x^2-5(x+\Delta x)^2}{x^2(x+\Delta x)^2}\right]$$

$$=\frac{5}{\Delta x}\left[\frac{x^2-\left(x^2+2x\Delta x-\Delta x^2\right)}{x^2(x+\Delta x)^2}\right]$$

$$=\frac{5}{\Delta x}\left[\frac{x^2-x^2-2x\Delta x+\Delta x^2}{x^2(x+\Delta x)^2}\right]$$

$$=\frac{5\left(-2x\Delta x+\Delta x^2\right)}{\Delta x\cdot x^2(x+\Delta x)^2}$$

$$=\frac{5\Delta x(-2x+\Delta x)}{\Delta x\cdot x^2(x+\Delta x)^2}$$

$$=\frac{5(-2x+\Delta x)}{x^2(x+\Delta x)^2}$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{5(-2x+\Delta x)}{x^2(x+\Delta x)^2}\right)$$

$$\frac{dy}{dx}=\frac{5(-2x+0)}{x^2(x+0)^2}=-\frac{10x}{x^2\cdot x^2}=-\frac{10}{x^3}$$

b) $\left.\frac{dy}{dx}\right|_{x=-1}=-\frac{10}{(-1)^3}=\frac{-10}{-1}=10$

and $\left.\frac{dy}{dx}\right|_{x=2}=\frac{-10}{(2)^3}=-\frac{10}{8}=-\frac{5}{4}$

In Exercises $(13-18)$, using limit approach, determine the derivative of f and determine values of x at which the slope of tangent equals to 0.

Solutions:

13. $f(x)=-3x^2+6x$

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\frac{-3(x+\Delta x)^2+6(x+\Delta x)-(-3x^2+6x)}{\Delta x}$$

$$=\frac{-3(x^2+2x\Delta x+\Delta x^2)+6(x+\Delta x)-(-3x^2+6x)}{\Delta x}$$

$$=\frac{-3x^2-6x\Delta x-3\Delta x^2+6x+6\Delta x+3x^2-6x}{\Delta x}$$

$$=\frac{-6x\Delta x-3\Delta x^2+6\Delta x}{\Delta x}$$

$$=\frac{\Delta x(-6x-3\Delta x+6)}{\Delta x}$$

$$=-6x-3\Delta x+6$$

Taking limit $\Delta x\to 0$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}(-6x-3\Delta x+6)$$

$$\frac{dy}{dx}=-6x-3(0)+6=-6x+6$$

Since slope is 0, so $\frac{dy}{dx}=0$

$$-6x+6=0$$

$$-6x=-6$$

$$x=1$$

14. $f(x)=-\frac{3}{x}$

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\frac{-\frac{3}{(x+\Delta x)}+\frac{3}{x}}{\Delta x}$$

$$=\frac{\frac{-3x+3(x+\Delta x)}{x(x+\Delta x)}}{\Delta x}$$

$$=\frac{-3x+3x+3\Delta x}{x\Delta x(x+\Delta x)}$$

$$=\frac{3\Delta x}{x\Delta x(x+\Delta x)}$$

$$=\frac{3}{x(x+\Delta x)}$$

Taking limit $\Delta x\to 0$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{3}{x(x+\Delta x)}\right)$$

$$\frac{dy}{dx}=\frac{3}{x(x+0)}=\frac{3}{x^2}$$

Since slope is 0, so $\frac{dy}{dx}=0$

$$\Rightarrow\frac{3}{x^2}=0$$

There is no value of x for which the equation is true.

15. $f(x)=cx^2$

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\frac{c(x+\Delta x)^2-cx^2}{\Delta x}$$

$$=\frac{cx^2+2cx\Delta x+c\Delta x^2-cx^2}{\Delta x}$$

$$=\frac{2cx\Delta x+c\Delta x^2}{\Delta x}$$

$$=\frac{\Delta x(2cx+c\Delta x)}{\Delta x}$$

$$=2cx+c\Delta x$$

Taking limit $\Delta x\to 0$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}(2cx+c\Delta x)$$

$$\frac{dy}{dx}=2cx+c(0)=2cx$$

Since slope is 0, so $\frac{dy}{dx}=0$

$$2cx=0\Rightarrow x=0$$

16. $f(x) = -4x^2 - 2x$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{-4(x+\Delta x)^2 - 2(x+\Delta x) - (-4x^2 - 2x)}{\Delta x}$$

$$= \frac{-4x^2 - 8x\Delta x - 4\Delta x^2 - 2x - 2\Delta x + 4x^2 + 2x}{\Delta x}$$

$$= \frac{-8x\Delta x - 4\Delta x^2 - 2\Delta x}{\Delta x}$$

$$= \frac{\Delta x(-8x - 4\Delta x - 2)}{\Delta x}$$

$$= -8x - 4\Delta x - 2$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (-8x - 4\Delta x - 2)$$

$$\frac{dy}{dx} = -8x - 4(0) - 2 = -8x - 2$$

Since slope is 0, so $\frac{dy}{dx} = 0$

$$-8x - 2 = 0 \quad \Rightarrow -8x = 2$$

$$\Rightarrow x = -\frac{2}{8} \quad \Rightarrow \quad x = -\frac{1}{4}$$

17. $f(x) = x^3 + x$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{(x+\Delta x)^3 + (x+\Delta x) - (x^3 + x)}{\Delta x}$$

$$= \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + x + \Delta x - x^3 - x}{\Delta x}$$

$$= \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + \Delta x}{\Delta x}$$

$$= \frac{\Delta x(3x^2 + 3x\Delta x + \Delta x^2 + 1)}{\Delta x}$$

$$= 3x^2 + 3x\Delta x + \Delta x^2 + 1$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + \Delta x^2 + 1)$$

$$\frac{dy}{dx} = 3x^2 + 3x(0) + (0)^2 + 1 = 3x^2 + 1$$

Since slope is 0, so $\frac{dy}{dx} = 0$

$$3x^2 + 1 = 0 \text{ (No real value)}$$

18. $f(x) = -10x^2 + 40x$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{-10(x+\Delta x)^2 + 40(x+\Delta x) - (-10x^2 + 40x)}{\Delta x}$$

$$= \frac{-10x^2 - 20x\Delta x - 10\Delta x^2 + 40x + 40\Delta x + 10x^2 - 40x}{\Delta x}$$

$$= \frac{-20x\Delta x - 10\Delta x^2 + 40\Delta x}{\Delta x}$$

$$= \frac{\Delta x(-20x - 10\Delta x + 40)}{\Delta x} = -20x - 10\Delta x + 40$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (-20x - 10\Delta x + 40)$$

$$\frac{dy}{dx} = -20x - 10(0) + 40 = -20x + 40$$

Since slope is 0, so $\frac{dy}{dx} = 0$

$$-20x + 40 = 0$$

$$-2x = -40 \Rightarrow x = \frac{-40}{-2} \Rightarrow x = 20$$

Exercises 4.3

In Exercise $(1-14)$, find the derivative w.r.t. x.

1. $y = x^4 - 3x^2$

Diff. w.r.t x

$$\frac{dy}{dx} = 4x^3 - 6x$$

2. $y = \sqrt[3]{x^7} \Rightarrow y = x^{\frac{7}{3}}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{7}{3}}\right)$$

$$\frac{dy}{dx} = \frac{7}{3} x^{\frac{7}{3}-1}$$

$$\frac{dy}{dx} = \frac{7}{3} x^{\frac{7-3}{3}} = \frac{7}{3} x^{\frac{4}{3}}$$

3. $y = \frac{x^4}{2} - 3x^2 + 10$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{4x^3}{2} - 6x = 2x^3 - 6x$$

4. $y = x^2 - \sqrt{x}$

Diff. w.r.t x

$$\frac{dy}{dx} = 2x - \frac{1}{2\sqrt{x}}$$

5. $y = (x^2 + 5)(x^3 + 7)$

$$y = x^5 + 7x^2 + 5x^3 + 35$$

$$y = x^5 + 5x^3 + 7x^2 + 35$$

Diff. w.r.t x

$$\frac{dy}{dx} = 5x^4 + 15x^2 + 14x$$

6. $y = x^{-3} + 2x^{-\frac{3}{2}} + 3$

Diff. w.r.t x

$$\frac{dy}{dx} = -3x^{-4} + 2\left(-\frac{3}{2}\right)x^{\frac{-3}{2}-1}$$

$$= -3x^{-4} - 3x^{\frac{-3-2}{2}}$$

$$= -3x^{-4} - 3x^{\frac{-5}{2}}$$

$$= -3\left(x^{-4} + x^{\frac{-5}{2}}\right)$$

7. $y = (2 - x - 3x^4)(10 + x - 4x^3)$

$$= 20 + 2x - 8x^3 - 10x - x^2 + 4x^4 - 30x^4 - 3x^5 + 12x^7$$

$$y = 12x^7 - 3x^5 - 26x^4 - 8x^3 - x^2 - 8x + 20$$

Diff. w.r.t x

$$\frac{dy}{dx} = 84x^6 - 15x^4 - 104x^3 - 24x^2 - 2x - 8$$

8. $y = \frac{3}{5x^3} \Rightarrow y = \frac{3}{5} \cdot x^{-3}$

Diff. w.r.t x

$$\frac{dy}{dx} = \left(\frac{3}{5}\right)\left(-3x^{-4}\right) = \frac{-9}{5x^4}$$

9. $y = \frac{10 - x}{x^2 + 2}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{\left(x^2+2\right)\frac{d}{dx}(10-x) - (10-x)\frac{d}{dx}\left(x^2+2\right)}{\left(x^2+2\right)^2}$$

$$= \frac{\left(x^2+2\right)(-1) - (10-x)(2x)}{\left(x^2+2\right)^2} = \frac{-x^2 - 2 - 20x + 2x^2}{\left(x^2+2\right)^2} = \frac{x^2 - 20x - 2}{\left(x^2+2\right)^2}$$

10. $y = \frac{x^2+1}{x^2-3}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{\left(x^2-3\right)\frac{d}{dx}\left(x^2+1\right) - \left(x^2+1\right)\frac{d}{dx}\left(x^2-3\right)}{\left(x^2-3\right)^2}$$

$$= \frac{\left(x^2-3\right)(2x) - \left(x^2+1\right)(2x)}{\left(x^2-3\right)^2}$$

$$= \frac{2x\left[\left(x^2-3\right) - \left(x^2+1\right)\right]}{\left(x^2-3\right)^2} = \frac{2x\left[x^2 - 3 - x^2 - 1\right]}{\left(x^2-3\right)^2} = \frac{2x(-4)}{\left(x^2-3\right)^2} \quad \Rightarrow \frac{dy}{dx} = \frac{-8x}{\left(x^2-3\right)^2}$$

11. $y = \frac{\left(x^2+1\right)(x-2)}{x}$

$$y = \frac{x^3 - 2x^2 + x - 2}{x} = \frac{x^3}{x} - \frac{2x^2}{x} + \frac{x}{x} - \frac{2}{x} = x^2 - 2x + 1 - \frac{2}{x}$$

$$y = x^2 - 2x + 1 - 2x^{-1}$$

Diff. w.r.t x

$$\frac{dy}{dx}=2x-2+2x^{-2}=2x-2+\frac{2}{x^2}\Rightarrow\frac{dy}{dx}=2\left(x-1+\frac{1}{x^2}\right)$$

12. $y=\dfrac{2x-1}{x^2+1}$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{\left(x^2+1\right)\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}$$

$$=\frac{\left(x^2+1\right)(2)-(2x-1)(2x)}{\left(x^2+1\right)^2}$$

$$=\frac{2x^2+2-4x^2+2x}{\left(x^2+1\right)^2}=\frac{-2x^2+2x+2}{\left(x^2+1\right)^2}\Rightarrow\frac{dy}{dx}=\frac{-2\left(x^2-x-1\right)}{\left(x^2+1\right)^2}$$

13. $y=\dfrac{3x^5}{x^2-2x+1}$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{\left(x^2-2x+1\right)\frac{d}{dx}\left(3x^5\right)-\left(3x^5\right)\frac{d}{dx}\left(x^2-2x+1\right)}{\left(x^2-2x+1\right)^2}$$

$$=\frac{\left(x^2-2x+1\right)\left(15x^4\right)-\left(3x^5\right)(2x-2)}{\left(x^2-2x+1\right)^2}=\frac{3x^4\left[\left(x^2-2x+1\right)(5)-x(2x-2)\right]}{\left(x^2-2x+1\right)^2}$$

$$=\frac{3x^4\left[5x^2-10x+5-2x^2+2x\right]}{\left(x^2-2x+1\right)^2}\Rightarrow\frac{dy}{dx}=\frac{3x^4\left(3x^2-8x+5\right)}{\left(x^2-2x+1\right)^2}$$

14. $y=\dfrac{x}{1-x^2}$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{\left(1-x^2\right)\frac{d}{dx}(x)-(x)\frac{d}{dx}\left(1-x^2\right)}{\left(1-x^2\right)^2}$$

$$=\frac{\left(1-x^2\right)(1)-x(-2x)}{\left(1-x^2\right)^2}$$

$$=\frac{1-x^2+2x^2}{\left(1-x^2\right)^2}$$

$$\frac{dy}{dx}=\frac{1+x^2}{\left(1-x^2\right)^2}$$

In Exercises $(15-20)$, find i) $f'(x)$ ii) $f'(1)$ iii) Determine the values of x for which $f'(x)=0$.

15. $f(x)=10x^2-5$

i) $f'(x) = 20x$

ii) $f'(1) = 20(1) = 20$

iii) $f'(x) = 0$

$20x = 0 \Rightarrow x = 0$

16. $f(x) = 8x^2 - 16x$

i) $f'(x) = 16x - 16$

ii) $f'(1) = 16(1) - 16 = 16 - 16 = 0$

iii) $f'(x) = 0$

$16x - 16 = 0 \Rightarrow 16x = 16 \Rightarrow x = 1$

17. $f(x) = \frac{x^3}{3} - 6x + 8$

i) $f'(x) = \frac{3x^2}{3} - 6 = x^2 - 6$

ii) $f'(1) = \frac{3(1)^2}{3} - 6 = 1 - 6 = -5$

iii) $f'(x) = 0$

$x^2 - 6 = 0 \Rightarrow x^2 = 6 \Rightarrow x = \pm\sqrt{6}$

18. $f(x) = -\frac{1}{x} = -x^{-1}$

i) $f'(x) = x^{-2} = \frac{1}{x^2}$

ii) $f'(1) = \frac{1}{(1)^2} = 1$

iii) $f'(x) = 0$

$\frac{1}{x^2} = 0 \Rightarrow 1 = 0$, No solution.

19. $f(x) = \frac{16x^4}{4 - x}$

i)

$$f'(x) = \frac{(4-x)\frac{d}{dx}(16x^4) - (16x^4)\frac{d}{dx}(4-x)}{(4-x)^2}$$

$$= \frac{(4-x)(64x^3) - (16x^4)(-1)}{(4-x)^2}$$

$$= \frac{16x^3[(4-x)(4) + x]}{(4-x)^2}$$

$$= \frac{16x^3[16 - 4x + x]}{(4-x)^2}$$

$$f'(x) = \frac{16x^3(16-3x)}{(4-x)^2}$$

ii) $f'(1) = \frac{16(1)^3(16-3(1))}{(4-(1))^2} = \frac{208}{9}$

iii) $f'(x) = 0$

$\frac{16x^3(16-3x)}{(4-x)^2} = 0$

$16x^3(16-3x) = 0$

$\Rightarrow \quad x = 0 \;,\; 16 - 3x = 0$

$3x = 16 \Rightarrow x = \frac{16}{3}$

20. $f(x) = x^2 - 16x + 5$

i) $f'(x) = 2x - 16$

ii) $f'(1) = 2(1) - 16 = 2 - 16 = -14$

iii) $f'(x) = 0$

$2x - 16 = 0. \Rightarrow 2x = 16 \Rightarrow x = 8$

Exercises 4.4

In Exercise $(1-28)$, determine $\frac{dy}{dx}$.

1. $y=(1-4x^2)^3$

Diff. w.r.t x

$$\frac{dy}{dx}=3(1-4x^2)^2\frac{d}{dx}(1-4x^2)$$

$$=3(1-4x^2)^2(-8x)$$

$$=-24x(1-4x^2)^2$$

2. $y=(7x^2-3x+5)^3$

Diff. w.r.t x

$$\frac{dy}{dx}=3(7x^2-3x+5)^2\frac{d}{dx}(7x^2-3x+5)$$

$$=3(7x^2-3x+5)^2(14x-3)$$

$$=3(14x-3)(7x^2-3x+5)^2$$

3. $y=(x^3-2x-1)^4$

Diff. w.r.t x

$$\frac{dy}{dx}=4(x^3-2x-1)^3\frac{d}{dx}(x^3-2x-1)$$

$$=4(x^3-2x-1)^3(3x^2-2)$$

$$=4(3x^2-2)(x^3-2x-1)^3$$

4. $y=\sqrt{2x^2+3}$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{1}{2\sqrt{2x^2+3}}\frac{d}{dx}(2x^2+3)$$

$$=\frac{1}{2\sqrt{2x^2+3}}\cdot(4x)$$

4. $y=e^{x^2+1}$

Diff. w.r.t x

$$\frac{dy}{dx}=e^{x^2+1}\frac{d}{dx}(x^2+1)$$

$$\frac{dy}{dx}=e^{x^2+1}(2x)=2xe^{x^2+1}$$

$$=\frac{2x}{\sqrt{2x^2+3}}$$

5. $y=\sqrt{x^2+2x+7}$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x^2+2x+7}}\frac{d}{dx}(x^2+2x+7)$$

$$=\frac{1}{2\sqrt{x^2+2x+7}}\cdot(2x+2)$$

$$=\frac{2(x+1)}{2\sqrt{x^2+2x+7}}$$

$$=\frac{(x+1)}{\sqrt{x^2+2x+7}}$$

6. $y=e^x$

Diff. w.r.t x

$$\frac{dy}{dx}=e^x$$

7. $y=e^{x^2+1}$

Diff. w.r.t x

$$\frac{dy}{dx}=e^{x^2+1}\frac{d}{dx}(x^2+1)$$

$$\frac{dy}{dx}=e^{x+1}(2x)=2xe^{x+1}$$

8. $y=-5e^{4x}$

Diff. w.r.t x

$$\frac{dy}{dx}=-5e^{4x}\frac{d}{dx}(4x)$$

$$=-5e^{4x}(4)=-20e^{4x}$$

9. $y=\left(e^x\right)^3$

Diff. w.r.t x

$$\frac{dy}{dx}=e^{3x}(3)=3\,e^{3x}$$

10. $y=\dfrac{2x^2}{e^x}$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{e^x\frac{d}{dx}\left(2x^2\right)-\left(2x^2\right)\frac{d}{dx}e^x}{\left(e^x\right)^2}$$

$$=\frac{e^x(4x)-\left(2x^2\right)e^x}{\left(e^x\right)^2}$$

$$=\frac{2xe^x(2-x)}{\left(e^x\right)^2}$$

$$=\frac{2x(2-x)}{e^x}$$

11. $y=\ln(5x)$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{1}{5x}\frac{d}{dx}(5x)$$

$$=\frac{1}{5x}\cdot(5)=\frac{1}{x}$$

12. $y=\ln\left(\dfrac{x}{5}\right)$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{1}{\frac{x}{5}}\frac{d}{dx}\left(\frac{x}{5}\right)$$

$$=\frac{5}{x}\cdot\left(\frac{1}{5}\right)=\frac{1}{x}$$

13. $y=\ln\left(x^2-3\right)$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{1}{x^2-3}\frac{d}{dx}\left(x^2-3\right)$$

$$=\frac{1}{x^2-3}(2x)=\frac{2x}{x^2-3}$$

14. $y=x^2\ln x$

Diff. w.r.t x

$$\frac{dy}{dx}=x^2\frac{d}{dx}(\ln x)+\ln x\cdot\frac{d}{dx}\left(x^2\right)$$

$$=x^2\cdot\frac{1}{x}+\ln x\cdot(2x)$$

$$=x(1+2\ln x)$$

15. $y=\ln\left(x^3-2x^2+5\right)$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{1}{x^3-2x^2+5}\frac{d}{dx}\left(x^3-2x^2+5\right)$$

$$=\frac{1}{x^3-2x^2+5}\cdot\left(3x^2-4x\right)$$

$$=\frac{3x^2-4x}{x^3-2x^2+5}$$

16. $y=(x+3)\cdot\ln x^2$

Diff. w.r.t x

$$\frac{dy}{dx}=(x+3)\frac{d}{dx}\left(\ln x^2\right)+\ln x^2\frac{d}{dx}(x+3)$$

$$=(x+3)\cdot\frac{1}{x^2}(2x)+\ln x^2\cdot(1)$$

$$=\frac{2(x+3)}{x}+\ln x^2$$

17. $y=(\ln x)^3$

Diff. w.r.t x

$$\frac{dy}{dx}=3(\ln x)^2\frac{d}{dx}(\ln x)$$

$$=3(\ln x)^2\cdot\left(\frac{1}{x}\right)$$

$$=\frac{3(\ln x)^2}{x}$$

18. $y=(2x-\ln x)^4$

Diff. w.r.t x

$$\frac{dy}{dx}=4(2x-\ln x)^3\ \frac{d}{dx}(2x-\ln x)$$

$$=4(2x-\ln x)^3\cdot\left(2-\frac{1}{x}\right)$$

$$=4(2x-\ln x)^3\cdot\left(\frac{2x-1}{x}\right)$$

$$=\frac{4(2x-1)(2x-\ln x)^3}{x}$$

19. $y=\ln\left(x^3+2x^2+4\right)$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{1}{x^3+2x^2+4}\ \frac{d}{dx}\left(x^3+2x^2+4\right)$$

$$=\frac{1}{x^3+2x^2+4}\cdot\left(3x^2+4x\right)$$

$$=\frac{3x^2+4x}{x^3+2x^2+4}$$

20. $y=\dfrac{10x}{\ln x}$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{\ln x\ \frac{d}{dx}(10x)-(10x)\frac{d}{dx}(\ln x)}{(\ln x)^2}$$

$$=\frac{\ln x\,(10)-(10x)\left(\frac{1}{x}\right)}{(\ln x)^2}$$

$$=\frac{10\ln x-10}{(\ln x)^2}$$

$$=\frac{10(\ln x-1)}{(\ln x)^2}$$

In Exercise $(21-25)$, Find $\dfrac{dy}{dx}$.

21.

$y=f(u)=5u+3\ ;\ u=g(x)=-3x+10$

Diff. w.r.t u Diff. w.r.t x

$$\frac{dy}{du}=\frac{d}{du}(5u+3)\quad,\quad\frac{du}{dx}=\frac{d}{dx}(-3x+10)$$

$$\frac{dy}{du}=5\quad,\quad\frac{du}{dx}=-3$$

By chain Rule

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=(5)\cdot(-3)=-15$$

22.

$$y=f(u)=u^2-3\quad;\quad u=g(x)=2x-1$$

Diff. w.r.t u Diff. w.r.t x

$$\frac{dy}{du}=\frac{d}{du}\left(u^2-3\right)\quad,\quad\frac{du}{dx}=\frac{d}{dx}(2x-1)$$

$$\frac{dy}{du}=2u\quad,\quad\frac{du}{dx}=2$$

By chain Rule

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=(2u)\cdot(2)$$

$$=4(2x-1)\qquad u=2x-1$$

23.

$$y=f(u)=u^3\quad;\quad u=g(x)=x^3+3x+1$$

Diff. w.r.t u Diff. w.r.t x

$$\frac{dy}{du}=3u^2\quad,\quad\frac{du}{dx}=3x^2+3$$

By chain Rule

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=\left(3u^2\right)\cdot\left(3x^2+3\right)$$

$$=9\left(x^2+1\right)u^2$$

$=4u$

$$=9\left(x^2+1\right)\left(x^2+3x+1\right)^2$$

24.

$y=f(u)=u^4-u^2+1 \quad ; \quad u=g(x)=x^2-4$

Diff. w.r.t u Diff. w.r.t x

$$\frac{dy}{du}=4u^3-2u \quad , \quad \frac{du}{dx}=2x$$

By chain Rule

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=\left(4u^3-2u\right)\cdot(2x)$$

$$=(2x)(2)\left(2u^3-u\right)$$

$$=4x\left[2\left(x^2-4\right)^3-\left(x^2-4\right)\right]$$

25. $y=f(u)=e^4 \quad ; \quad u=g(x)=\sqrt{x}$

Diff. w.r.t u Diff. w.r.t x

$$\frac{dy}{du}=e^4 \quad , \quad \frac{du}{dx}=\frac{1}{2\sqrt{x}}$$

By chain Rule

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=\left(e^4\right)\cdot\left(\frac{1}{2\sqrt{x}}\right)=\frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

Exercises 4.5

In Exercises $(1-16)$, Find the derivative.

Solutions:

1. $f(x)=4\cos x$

Diff. w.r.t x

$$f'(x)=4(-\sin x)=-4\sin x$$

2. $g(x)=7\tan x$

Diff. w.r.t x

$$g'(x)=7\left(\sec^2 x\right)$$

3. $f(u)=3u\sin u$

Diff. w.r.t x

$$f'(u)=3u\frac{d}{dx}(\sin u)+\sin u\frac{d}{dx}(3u)$$

$$=(3u)(\cos u)+(\sin u)(3)$$

$$=3(u\cos u+\sin u)$$

4. $f(x)=2x\cdot\tan x$

Diff. w.r.t x

$$f'(x)=2x\frac{d}{dx}(\tan x)+\tan x\frac{d}{dx}(2x)$$

$$=(2x)\left(\sec^2 x\right)+(\tan x)(2)$$

$$=2\left(x\sec^2 x+\tan x\right)$$

5. $g(t)=t^2-\sec t$

Diff. w.r.t t

$$g'(t)=2t-\sec t\tan t$$

6. $f(x)=x^2+x\sin x$

Diff. w.r.t x

$$f'(x)=2x+x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x)$$

$$=2x+x\cos x+\sin x\,(1)$$

$$=2x+\sin x+x\cos x$$

7. $g(x)=2x\cos x+x^2\sin x$

Diff. w.r.t x

$$g'(x)=\left[2x\frac{d}{dx}(\cos x)+\cos x\frac{d}{dx}(2x)\right]+\left[x^2\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x^2)\right]$$

$$=\left[2x(-\sin x)+\cos x\,(2)\right]+\left[x^2(\cos x)+\sin x(2x)\right]$$

$$=-2x\sin x+2\cos x+x^2\cos x+2x\sin x$$

$$=2\cos x+x^2\cos x$$

$$g'(x)=\cos x\left(2+x^2\right)$$

8. $g(x)=\dfrac{\cos x}{1-\sin x}$

Diff. w.r.t x

$$g'(x)=\frac{(1-\sin x)\frac{d}{dx}(\cos x)-\cos x\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$

$$=\frac{(1-\sin x)(-\sin x)-\cos x(-\cos x)}{(1-\sin x)^2}$$

$$=\frac{-\sin x+\sin^2 x+\cos^2 x}{(1-\sin x)^2}=\frac{1-\sin x}{(1-\sin x)^2}=\frac{1}{1-\sin x}$$

9. $f(x)=(x+\cos x)\tan x$

Diff. w.r.t x

$$f'(x)=(x+\cos x)\frac{d}{dx}(\tan x)+\tan x\frac{d}{dx}(x+\cos x)$$

$$=(x+\cos x)\left(\sec^2 x\right)+\tan x(1-\sin x)$$

$$=\sec^2 x(x+\cos x)+\tan x(1-\sin x)$$

10. $f(x)=\sin x\operatorname{cosec} x$

$$f(x)=\frac{\sin x}{\sin x}=1$$

Diff. w.r.t x

$$f'(x)=0$$

11. $f(x)=\sin\left(2x^3\right)$

Diff. w.r.t x

$$f'(x)=\cos\left(2x^3\right)\frac{d}{dx}\left(2x^3\right)$$

$$=\cos\left(2x^3\right)\left(6x^2\right)$$

$$=6x^2\cos\left(2x^3\right)$$

12. $f(x)=\sin x\cot x$

$$=\sin x\cdot\frac{\cos x}{\sin x}$$

$$=\cos x$$

Diff. w.r.t x

$f'(x)=-\sin x$

13. $f(t)=\cos\left(\sqrt{t}\right)$

Diff. w.r.t t

$$f'(t)=-\sin\left(\sqrt{t}\right)\frac{d}{dx}\left(\sqrt{t}\right)$$

$$=\frac{-\sin\left(\sqrt{t}\right)}{2\sqrt{t}}$$

14. $g(t)=t^2\cdot\sec 4t$

Diff. w.r.t t

$$g'(t)=t^2\frac{d}{dt}(\sec 4t)+\sec 4t\cdot\frac{d}{dt}(t^2)$$

$$g'(t)=t^2(\sec 4t\cdot\tan 4t)(4)+\sec 4t(2t)$$

$$=2t\sec 4t(2t\tan 4t+1)$$

15. $f(x)=(\sin 2x-\cos 3x)^2$

Diff. w.r.t x

$$f'(x)=2(\sin 2x-\cos 3x)\frac{d}{dx}(\sin 2x-\cos 3x)$$

$$=2(\sin 2x-\cos 3x)\cdot(\cos 2x(2)+\sin 3x(3))$$

$$=2(\sin 2x-\cos 3x)\cdot(2\cos 2x+3\sin 3x)$$

16. $f(x)=\sqrt{\tan x}$

Diff. w.r.t x

$$f'(x)=\frac{1}{2\sqrt{\tan x}}\frac{d}{dx}(\tan x)$$

$$=\frac{\sec^2 x}{2\sqrt{\tan x}}$$

In Exercises $(17-20)$, Find $f'\left(\frac{\pi}{2}\right)$, $f'\left(\frac{\pi}{4}\right)$.

17. $f(x)=\sin 2x$

Diff. w.r.t x

$$f'(x)=\cos 2x\,(2)=2\cos 2x$$

$$f'\left(\frac{\pi}{2}\right)=2\cos\left(2\cdot\frac{\pi}{2}\right)$$

$$=2\cos\pi=2(-1)=-2$$

$$f'\left(\frac{\pi}{4}\right)=2\cos\left(2\cdot\frac{\pi}{4}\right)$$

$$=2\cos\frac{\pi}{2}=2(0)=0$$

18. $f(x)=\cos x$

Diff. w.r.t x

$f'(x)=-\sin x$

$$f'\left(\frac{\pi}{2}\right)=-\sin\frac{\pi}{2}=-1$$

$$f'\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

19. $f(x)=\sin^2 x$

Diff. w.r.t x

$f'(x)=2\sin x\cdot\cos x$

$$f'\left(\frac{\pi}{2}\right)=2\sin\frac{\pi}{2}\cos\frac{\pi}{2}=2(1)(0)=0$$

$$f'\left(\frac{\pi}{4}\right)=2\sin\frac{\pi}{4}\cos\frac{\pi}{4}$$

$$=2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)=\frac{2}{2}=1$$

20. $f(x)=\tan x$

Diff. w.r.t x

$f'(x)=\sec^2 x$

$$f'\left(\frac{\pi}{2}\right)=\sec^2\left(\frac{\pi}{2}\right)=\frac{1}{\cos^2\frac{\pi}{2}}=\frac{1}{0}=\infty$$

$$f'\left(\frac{\pi}{4}\right)=\sec^2\left(\frac{\pi}{4}\right)=\left(\sqrt{2}^2\right)=2$$

Exercises 4.6

In Exercises $(1-10)$, Find $f''(x)$.

Solutions:

1. $f(x)=2x^5-3x^4+4x^3+x-2$

Diff. w.r.t x

$f'(x)=10x^4-12x^3+12x^2+1$

Diff. again

$f''(x)=40x^3-36x^2+24x$

2. $f(x)=(2x+5)^{\frac{3}{2}}$

Diff. w.r.t x

$$f'(x)=\frac{3}{2}(2x+5)^{\frac{3}{2}-1}\frac{d}{dx}(2x+5)$$

$$=\frac{3}{2}(2x+5)^{\frac{1}{2}}(2)$$

$$=3(2x+5)^{\frac{1}{2}}$$

Diff. again

$$f''(x)=3\cdot\frac{1}{2}(2x+5)^{\frac{1}{2}-1}$$

$$=\frac{3}{2}(2x+5)^{-\frac{1}{2}}=\frac{3}{2\sqrt{2x+5}}$$

3. $f(x)=\frac{x^4}{4}-\frac{x^3}{3}+10x$

Diff. w.r.t x

$$f'(x)=\frac{4x^3}{4}-\frac{3x^2}{3}+10$$

$f'(x)=x^3-x^2+10$

Diff. again

$f''(x)=3x^2-2x$

4. $f(x)=-\frac{4}{x^3}$

$f(x)=-\frac{4}{x^3}$

$f(x)=-4x^{-3}$

Diff. w.r.t x

$f'(x)=12x^{-4}$

Diff. again

$f''(x)=-48x^{-5}=-\frac{48}{x^5}$

5. $f(x)=\frac{2}{x^2}$

Diff. w.r.t x

$f'(x)=-4x^{-3}$

Diff. again

$f''(x)=12x^{-4}=\frac{12}{x^4}$

6. $f(x)=\sqrt{x+1}$

Diff. w.r.t x

$$f'(x)=\frac{1}{2\sqrt{x+1}}=2(x+1)^{\frac{-1}{2}}$$

Diff. again

$$f''(x)=(2)\left(-\frac{1}{2}\right)(x+1)^{-\frac{1}{2}-1}$$

$$=-(x+1)^{-\frac{3}{2}}=-\frac{1}{(x+1)^{\frac{3}{2}}}$$

7. $f(x)=e^{x^2}$

Diff. w.r.t x

$$f'(x)=e^{x^2}\frac{d}{dx}(x^2)$$

Diff. again

$$f''(x)=e^{x^2}\frac{d}{dx}(2x)+2x\frac{d}{dx}\left(e^{x^2}\right)$$

$$=e^{x^2}(2)+2x\cdot\left(e^{x^2}\right)(2x)$$

$$=e^{x^2}\left(1+2x^2\right)$$

8. $f(x)=\ln 2x$

Diff. w.r.t x

$$f'(x)=\frac{1}{2x}\frac{d}{dx}(2x)$$

$$=\frac{2}{2x}=x^{-1}$$

Diff. again

$$f''(x)=-x^{-2}=-\frac{1}{x^2}$$

9. $f(x)=\ln(x^2+4)$

Diff. w.r.t x

$$f'(x)=\frac{1}{x^2+4}\frac{d}{dx}(x^2+4)$$

$$=\frac{2x}{x^2+4}$$

Diff. again

$$f''(x)=\frac{(x^2+4)\frac{d}{dx}(2x)-(2x)\frac{d}{dx}(x^2+4)}{(x^2+4)^2}$$

$$=\frac{(x^2+4)(2)-(2x)(2x)}{(x^2+4)^2}$$

$$=\frac{2x^2+8-4x^2}{(x^2+4)^2}$$

$$=\frac{8-2x^2}{(x^2+4)^2}=\frac{2(4-x^2)}{(x^2+4)^2}$$

10. $f(x)=\ln(x^2-5)$

Diff. w.r.t x

$$f'(x)=\frac{1}{x^2-5}\frac{d}{dx}(x^2-5)$$

$$=\frac{1}{x^2-5}(2x)=\frac{2x}{x^2-5}$$

Diff. again

$$f''(x)=\frac{(x^2-5)\frac{d}{dx}(2x)-(2x)\frac{d}{dx}(x^2-5)}{(x^2-5)^2}$$

$$=\frac{(x^2-5)(2)-(2x)(2x)}{(x^2-5)^2}$$

$$=\frac{2x^2-10-4x^2}{(x^2-5)^2}$$

$$=\frac{-2x^2-10}{(x^2-5)^2}=\frac{-2(x^2+5)}{(x^2-5)^2}$$

In Exercises $(11-16)$, Find $\frac{d^4y}{dx^4}$.

11. $y=\sin 3x$

$$\frac{dy}{dx}=\cos 3x(3)=3\cos 3x$$

$$\Rightarrow \frac{d^2y}{dx^2}=3(-\sin 3x)(3)=-9\sin 3x$$

$$\Rightarrow \frac{d^3y}{dx^3}=-9(\cos 3x)(3)=-27\cos 3x$$

$$\Rightarrow \frac{d^4y}{dx^4}=-27(-\sin 3x)(3)=81\sin 3x$$

12. $y=\cos 2x$

$$\frac{dy}{dx}=-\sin 2x(2)=-2\sin 2x$$

$$\Rightarrow \frac{d^2y}{dx^2}=-2(\cos 2x)(2)=-4\cos 2x$$

$$\Rightarrow \frac{d^3y}{dx^3}=-4(-\sin 2x)(2)=8\sin 2x$$

$$\Rightarrow \frac{d^4y}{dx^4}=8(\cos 2x)(2)=16\cos 2x$$

13. $y=\ln(x^2-9)$

$$y=\ln(x+3)(x-3)$$

$$y=\ln(x+3)\ln(x-3)$$

$$\frac{dy}{dx}=\frac{1}{x+3}(1)+\frac{1}{x-3}(1)$$

$$\frac{dy}{dx}=(x+3)^{-1}+(x-3)^{-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(x+3)^{-2} - (x-3)^{-2}$$

$$\Rightarrow \frac{d^3y}{dx^3} = 2(x+3)^{-3} + 2(x-3)^{-3}$$

$$\Rightarrow \frac{d^4y}{dx^4} = -6(x+3)^{-4} - 6(x-3)^{-4}$$

$$= -6\left[(x+3)^{-4} + (x-3)^{-4}\right]$$

$$= -6\left[\frac{1}{(x+3)^4} + \frac{1}{(x-3)^4}\right]$$

14. $y = e^{-x}$

$$\frac{dy}{dx} = -e^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^{-x}$$

$$\Rightarrow \frac{d^3y}{dx^3} = -e^{-x}$$

$$\Rightarrow \frac{d^4y}{dx^4} = e^{-x}$$

15. $y = xe^{2x}$

$$\frac{dy}{dx} = x\frac{d}{dx}(e^{2x}) + e^{2x}\frac{d}{dx}(x)$$

$$= x\,e^{2x}(2) + e^{2x}(1)$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^{2x}\frac{d}{dx}(2x+1) + (2x+1)\frac{d}{dx}(e^{2x})$$

$$= e^{2x}(2) + (2x+1)(2e^{2x})$$

$$= 2e^{2x}(1+2x+1)$$

$$= 2e^{2x}(2x+2)$$

$$= 4e^{2x}(x+1)$$

$$\Rightarrow \frac{d^3y}{dx^3} = 4e^{2x}\frac{d}{dx}(x+1) + (x+1)\frac{d}{dx}(4e^{2x})$$

$$= 4e^{2x}\cdot(1) + (x+1)(8e^{2x})$$

$$= 4e^{2x}(1+2x+2)$$

$$= 4e^{2x}(2x+3)$$

$$\Rightarrow \frac{d^4y}{dx^4} = 4e^{2x}\frac{d}{dx}(2x+3) + (2x+3)\frac{d}{dx}(4e^{2x})$$

$$= 4e^{2x}(2) + (2x+3)(8e^{2x})$$

$$= 8e^{2x} + 8e^{2x}(2x+3)$$

$$= 8e^{2x}(1+2x+3)$$

$$= 8e^{2x}(2x+4)$$

$$= 16e^{2x}(x+2)$$

16. $y = e^{\frac{x}{2}}$

$$\frac{dy}{dx} = e^{\frac{x}{2}}\frac{d}{dx}\left(\frac{x}{2}\right) = e^{\frac{x}{2}}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}e^{\frac{x}{2}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{4}e^{\frac{x}{2}}$$

$$\Rightarrow \frac{d^3y}{dx^3} = \frac{1}{8}e^{\frac{x}{2}}$$

$$\Rightarrow \frac{d^4y}{dx^4} = \frac{1}{16}e^{\frac{x}{2}}$$

In Exercise $(17-24)$, Find all higher order derivatives.

Solutions:

17. $f(x) = 500$

$$f'(x) = 0$$

18. $f(x) = 16x^2 - 4x$

$$\Rightarrow f'(x) = 32x - 4$$

$$\Rightarrow f''(x) = 32$$

$$\Rightarrow f'''(x) = 0$$

19. $f(x) = mx + c$

$$\Rightarrow f'(x) = m$$

$$\Rightarrow f''(x) = 0$$

20. $f(x)=-\frac{x}{4}+10$

$\Rightarrow f'(x)=-\frac{1}{4}$

$\Rightarrow f''(x)=0$

21. $f(x)=10x^4-2x^3+3$

$\Rightarrow f'(x)=40x^3-6x^2$

$\Rightarrow f''(x)=120x^2-12x$

$\Rightarrow f'''(x)=240x-12$

$\Rightarrow f^{(iv)}(x)=240$

$\Rightarrow f^{(v)}(x)=0$

22. $f(x)=-x^5+3x^2$

$\Rightarrow f'(x)=-5x^4+6x$

$\Rightarrow f''(x)=-20x^3+6$

$\Rightarrow f'''(x)=-60x^2$

$\Rightarrow f^{(iv)}(x)=-120x$

$\Rightarrow f^{(v)}(x)=-120$

$\Rightarrow f^{(vi)}(x)=0$

23. $f(x)=e^x$

$\Rightarrow f'(x)=e^x$

$\Rightarrow f''(x)=e^x$

We see that all higher order derivatives are e^x.

24. $f(x)=\frac{x^4}{4}-\frac{x^3}{3}$

$\Rightarrow f'(x)=\frac{4x^3}{4}-\frac{3x^2}{3}$

$=x^3-x^2$

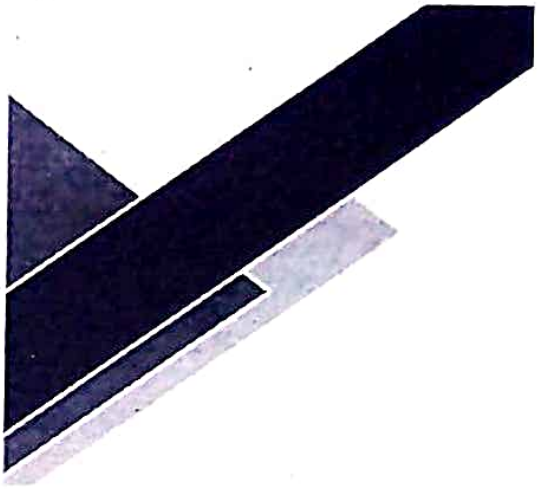
$\Rightarrow f''(x)=3x^2-2x$

$\Rightarrow f'''(x)=6x-2$

$\Rightarrow f^{(iv)}(x)=6$

$\Rightarrow f^{(v)}(x)=0$

CHAPTER 5
APPLICATIONS OF DERIVATIVES



Exercises 5.1

In Exercises $(1-8)$, identify the intervals on which f is increasing or decreasing.

1. $f(x)=16-x^2$

Diff.. w.r.t x

$f'(x)=-2x$

Clearly $f'(x)=0$ gives $-2x=0\Rightarrow x=0$

$\Rightarrow x=0$ is the stationary point.

We see that $f'(x)<0$ if $x>0$, so f is decreasing in the interval $(0,+\infty)$.

Also $f'(x)>0$ if $x<0$, so f is increasing in the interval $(-\infty,0)$.

2. $f(x)=x^2-9$

Diff ..w.r.t x

$f'(x)=2x$

Clearly $f'(x)=0$ gives $2x=0\Rightarrow x=0$

$\Rightarrow x=0$ is the stationary point.

We see that $f'(x)>0$ if $x>0$, so f is increasing in the interval $(0,+\infty)$.

Also $f'(x)<0$ if $x<0$, so f is decreasing in the interval $(-\infty,0)$.

3. $f(x)=x^2-4x$

Diff.. w.r.t x

$f'(x)=2x-4$

Clearly $f'(x)=0$ gives

$2x-4=0\Rightarrow x=2$

$\Rightarrow x=2$ is the stationary point.

We see that $f'(x)>0$ if $x>2$, so f is increasing in the interval $(2,+\infty)$

Also $f'(x)<0$ if $x<2$, so f is decreasing in the interval $(-\infty,2)$.

4. $f(x)=4x-x^2$

Diff .. w.r.t x

$f'(x)=4-2x$

Clearly $f'(x)=0$ gives $4-2x=0\Rightarrow x=2$

$\Rightarrow x=2$ is the stationary point.

We see that $f'(x)>0$ if $x<2$, so f is increasing in the interval $(-\infty,2)$

Also $f'(x)<0$ if $x>2$, so f is decreasing in the interval $(2,+\infty)$.

5. $f(x)=x^3-12x-5$

Diff .. w.r.t x

$f'(x)=3x^2-12$

Clearly $f'(x)=0$ gives $3x^2-12=0$

$3x^2=12 \Rightarrow x^2=4 \Rightarrow x=\pm 2$

The interval $(-\infty,+\infty)$ is subdivided into subintervals $(-\infty,-2)$, $(-2,2)$ and $(2,+\infty)$.

We see that $f'(x)>0$ if $x<-2$, so f is increasing in the interval $(-\infty,-2)$

$f'(x)<0$ if $x\in(-2,2)$, so f is decreasing in the interval $(-2,2)$

Also $f'(x)>0$ if $x>2$, so f is increasing in the interval $(2,+\infty)$.

6. $f(x)=x^2-3x+1$

Diff .. w.r.t x

$f'(x)=2x-3$

Clearly $f'(x)=0$ gives $2x-3=0 \Rightarrow x=\frac{3}{2}$

$\Rightarrow x=\frac{3}{2}$ is the stationary point.

We see that $f'(x)>0$ if $x>\frac{3}{2}$, so f is increasing in the interval $\left(\frac{3}{2},+\infty\right)$.

Also $f'(x)<0$ if $x<\frac{3}{2}$, so f is decreasing in the interval $\left(-\infty,\frac{3}{2}\right)$

7. $f(x)=(x-1)^2$

Diff .. w.r.t x

$f'(x)=2(x-1)$

Clearly $f'(x)=0$ gives

$2(x-1)=0 \Rightarrow x=1$

$\Rightarrow x=1$ is the stationary point.

We see that $f'(x)>0$ if $x>1$, so f is increasing in the interval $(1,+\infty)$

Also $f'(x)<0$ if $x<1$, so f is decreasing in the interval $(-\infty,1)$

8. $f(x)=12x-x^3$

Diff.. w.r.t x

$f'(x)=12-3x^2$

Clearly $f'(x)=0$ gives $12-3x^2=0$

$12=3x^2$

$x^2=4 \Rightarrow x=\pm 2$

The interval $(+\infty,+\infty)$ is subdivided in to subintervals $(-\infty,-2)$, $(-2,2)$ and $(2,+\infty)$

We see that $f'(x)<0$ if $x<-2$ and $x>2$,so f is decreasing in the interval $(-\infty,-2)$ and $(2,+\infty)$.

Also $f'(x)>0$ if $x\in(-2,2)$, so f is Increasing in the interval $(-2,2)$.

In Exercises $(9-26)$, determine the intervals on which f is increasing or decreasing g for the domain mentioned in each case.

9. $f(x)=4-x^2 \quad ; \quad x\in(-2,2)$

$f(x)=4-x^2$

Diff.. w.r.t x

$f'(x)=-2x$

Clearly $f'(x)=0$ gives $-2x=0 \Rightarrow x=0$

$\Rightarrow x=0$ is the stationary point.

The interval $(-2,2)$ is subdivided in the subintervals $(-2,0)$ and $(0,2)$.

We see that $f'(x)>0$ if $x\in(-2,0)$, so f is increasing in the interval $(-2,0)$

Also $f'(x)<0$ if $x\in(0,2)$, so f is decreasing in the interval $(0,2)$.

10. $f(x)=x^2+3x+2 \quad ; \quad x\in(-4,1)$

$f(x)=x^2+3x+2$

Diff.. w.r.t x

$f'(x)=2x+3$

Clearly $f'(x)=0$ gives

$2x+3=0\Rightarrow x=-\frac{3}{2}$

$\Rightarrow x=-\frac{3}{2}$ is the stationary point.

The interval $(-4,1)$ is subdivided into subintervals $\left(-4,-\frac{3}{2}\right)$ and $\left(-\frac{3}{2},1\right)$

We see that $f'(x)<0$ if $x\in\left(-4,-\frac{3}{2}\right)$, so f is decreasing in the interval $\left(-4,-\frac{3}{2}\right)$.

Also $f'(x)>0$ if $x\in\left(-\frac{3}{2},1\right)$, so f is increasing in the interval $\left(-\frac{3}{2},1\right)$.

11. $f(x)=-x^2-6x-9$; $x\in(-4,0)$

$f(x)=-x^2-6x-9$

Diff.. w.r.t x

$f'(x)=-2x-6$

Clearly $f'(x)=0$ gives $-2x-6=0$

$\Rightarrow -2x=6\Rightarrow x=-3$

$\Rightarrow x=-3$ is the stationary point.

The interval $(-4,0)$ is subdivided into subintervals $(-4,-3)$ and $(-3,0)$.

We see that $f'(x)>0$ if $x\in(-4,-3)$, so f is increasing in the interval $(-4,-3)$.

Also $f'(x)<0$ if $x\in(-3,0)$, so f is decreasing in the interval $(-3,0)$.

12. $f(x)=x^2-4x+4$

Diff.. w.r.t x

$f'(x)=2x-4$

Clearly $f'(x)=0$ gives $2x-4=0$

$\Rightarrow x=2$

$\Rightarrow x=2$ is the stationary point.

The interval $(1,3)$ is subdivided into subintervals $(1,2)$ and $(2,3)$.

We see that $f'(x)<0$ if $x\in(1,2)$, so f is decreasing in the interval $(1,2)$.

Also $f'(x)>0$ if $x\in(2,3)$, so f is increasing in the interval $(2,3)$.

13. $f(x)=x^3-3x^2$; $x\in(-3,3)$

$f(x)=x^3-3x^2$

Diff.. w.r.t x

$f'(x)=3x^2-6x$

Clearly $f'(x)=0$ gives $3x^2-6x=0$

$\Rightarrow 3x(x-2)=0\Rightarrow x=0,2$

The interval $(-3,3)$ is subdivided into subintervals $(-3,0)$ $(0,2)$ and $(2,3)$.

We see that $f'(x)>0$ if $x\in(-3,0)$ and $(2,3)$, so f is increasing in the interval $(-3,0)$ and $(2,3)$.

Also $f'(x)<0$ if $x\in(0,2)$, so f is

decreasing in the interval $(0,2)$.

14. $f(x)=2x-x^2 \quad ; \quad x\in(-2,2)$

Diff.. w.r.t x

$f'(x)=2-2x$

Clearly $f'(x)=0$ gives

$2-2x=0\Rightarrow 2x=2\Rightarrow x=1$

$\Rightarrow x=1$ is the stationary point.

The interval $(-2,2)$ is subdivided in to subintervals $(-2,1)$ and $(1,2)$.

We see that $f'(x)>0$ if $x\in(-2,1)$, so f is increasing in the interval $(-2,1)$.

Also $f'(x)<0$ if $x\in(1,2)$, so f is decreasing in the interval $(1,2)$.

15. $f(x)=3x^2-4x+1 \; ; x\in(0,2)$

$f(x)=3x^2-4x+1$

Diff. w.r.t x

$f'(x)=6x-4$

Clearly $f'(x)=0$ gives

$6x-4=0\Rightarrow x=\frac{2}{3}$

$\Rightarrow x=\frac{2}{3}$ is the stationary point.

The interval $(0,2)$ is subdivided in to subintervals $\left(0,\frac{2}{3}\right)$ and $\left(\frac{2}{3},2\right)$.

We see that $f'(x)<0$ if $x\in\left(0,\frac{2}{3}\right)$, so f is decreasing in the interval $\left(0,\frac{2}{3}\right)$.

Also $f'(x)>0$ if $x\in\left(\frac{2}{3},2\right)$, so f is increasing in the interval $\left(\frac{2}{3},2\right)$.

16. $f(x)=3x^2+12x+9 \; ; \; x\in(-3,3)$

Diff.. w.r.t x

$f'(x)=6x+12$

Clearly $f'(x)=0$ gives

$6x+12=0\Rightarrow 6x=-12\Rightarrow x=-2$

$\Rightarrow x=-2$ is the stationary point.

The interval $(-3,3)$ is subdivided into subintervals $(-3,-2)$ and $(-2,3)$.

We see that $f'(x)<0$ if $x\in(-3,-2)$, so f is decreasing in the interval $(-3,-2)$.

Also $f'(x)>0$ if $x\in(-2,3)$, so f is increasing in the interval $(-2,3)$.

17. $f(x)=-2x^2-12x-4; \quad x\in(-\infty,\infty)$

Diff .. w.r.t x

$f'(x)=-4x-12$

Clearly $f'(x)=0$ gives

$-4x-12=0\Rightarrow x=-3$

$\Rightarrow x=-3$ is the stationary point.

We see that $f'(x)<0$ if $x>-3$, so f is decreasing in the interval $(-3,+\infty)$.

Also $f'(x)>0$ if $x<-3$, so f is increasing in the interval $(-\infty,-3)$.

18. $f(x)=x^2-6x; \quad x\in(-\infty,\infty)$

Diff .. w.r.t x

$f'(x)=2x-6$

Clearly $f'(x)=0$ gives

$2x-6=0\Rightarrow x=3$

$\Rightarrow x=3$ is the stationary point.

We see that $f'(x)>0$ if $x>3$, so f is increasing in the interval $(3,+\infty)$.

Also $f'(x)<0$ if $x<3$, so f is decreasing in the interval $(-\infty,3)$.

19. $f(x)=(x+2)^2$; $x\in(-\infty,0)$

Diff .. w.r.t x

$f'(x)=2(x+2)$

Clearly $f'(x)=0$ gives

$2(x+2)=0\Rightarrow x=-2$

$\Rightarrow x=-2$ is the stationary point.

The interval $(-\infty,0)$ is subdivided into subintervals $(-\infty,-2)$ and $(-2,0)$.

We see that $f'(x)<0$ if $x\in(-\infty,-2)$, so f is decreasing in the interval $(-\infty,-2)$.

Also $f'(x)>0$ if $x\in(-2,0)$, so f is increasing in the interval $(-2,0)$.

20. $f(x)=x^3-12x$; $x\in(-\infty,\infty)$

Diff .. w.r.t x

$f'(x)=3x^2-12$

Clearly $f'(x)=0$ gives

$3x^2-12=0\Rightarrow x^2=4\Rightarrow x=\pm2$

$\Rightarrow x=-2,2$ are the stationary points.

The interval is subdivided into subintervals $(-\infty,-2),(-2,2)$ and $(2,+\infty)$.

We see that $f'(x)>0$ if $x\in(-\infty,-2)$ and $x\in(2,+\infty)$, so f is increasing in the intervals $(-\infty,-2)$ and $(2,+\infty)$.

Also $f'(x)<0$ if $x\in(-2,2)$. so f is decreasing in the interval $(-2,2)$.

Exercises 5.2

In Exercises $(1-6)$, find all the critical points:

1. $f(x)=2x^3-3x^2$

$f'(x)=6x^2-6x$

For critical points, $f'(x)=0$

$6x^2-6x=0$

$6x(x-1)=0\Rightarrow x=0,1$

Hence critical points are 0,1.

2. $f(x)=4x^3+6x^2-24x$

$f'(x)=12x^2+12x-24$

For critical points, $f'(x)=0$

$12x^2+12x-24=0$

$12(x^2+x-2)=0$

$x^2+x-2=0$

$x^2+2x-x-2=0$

$x(x+2)-1(x+2)=0$

$(x-1)(x+2)=0$

$x-1=0$, $x+2=0$

$x=1$, $x=-2\Rightarrow x=1,-2$

Hence critical points are 1,-2.

3. $f(x)=(x-1)^{\frac{2}{3}}$

$f'(x)=\frac{2}{3}(x-1)^{\frac{2}{3}-1}$

$=\frac{2}{3}(x-1)^{-\frac{1}{2}}=\frac{2}{3\sqrt{x-1}}$

We see that $f'(x)$ is undefined at $x=1$

Therefore, the critical point is 1.

4. $f(x)=3x^4+12x$

$f'(x)=12x^3+12$

For critical points, $f'(x)=0$

$12x^3+12=0\Rightarrow x^3+1=0$

$\Rightarrow(x+1)(x^2-x+1)=0$

$\Rightarrow x+1=0\Rightarrow x=-1$

$x^2 - x + 1 = 0$, gives imaginary values.

Hence critical point is -1.

5. $f(x) = 4x^4 - 16x^2 + 17$

$f'(x) = 16x^3 - 32x$

For critical points, $f'(x) = 0$

$16x^3 - 32x = 0$

$16x(x^2 - 2) = 0$

$x = 0,\ x^2 - 2 = 0$

$x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

Hence critical points are $-\sqrt{2}, 0, \sqrt{2}$.

6. $f(x) = (x-1)^2$

$f'(x) = 2(x-1)$

For critical points, $f'(x) = 0$

$2(x-1) = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

Hence critical point is 1.

In Exercise $(7-18)$, find the absolute maximum and minimum values of each function on the given interval.

7. $f(x) = \frac{2}{3}x - 5 \quad , \quad x \in [-2, 3]$

$f'(x) = \frac{2}{3}$

We see that there is no critical point.

The functional values at the end points $x = -2$ and $x = 3$ are

$f(-2) = \frac{2}{3}(-2) - 5 = -\frac{4}{3} - 5$

$= \frac{-4-15}{3} = -\frac{19}{3}$

$f(3) = \frac{2}{3}(3) - 5 = 2 - 5 = -3$

Thus $M = -3$ at $x = 3$ and $m = -\frac{19}{3}$ at $x = -2$.

8. $f(x) = -x - 4 \quad , \quad x \in [-4, 1]$

$f'(x) = -1$

We see that there is no critical point.

The functional values at the end points $x = -4$ and $x = 1$ are

$f(-4) = -(-4) - 4 = 4 - 4 = 0$

$f(1) = -1 - 4 = -5$

Thus $M = 0$ at $x = -4$ and $m = -5$ at $x = 1$

9. $f(x) = x^2 - 1 \quad , \quad x \in [-1, 2]$

$f'(x) = 2x$

For critical points, $f'(x) = 0$

$2x = 0$

$\Rightarrow x = 0$

We check that the functional values at $x = 0$ and at the end points $x = -1$ and $x = 2$.

Critical point value:

$f(0) = (0)^2 - 1 = -1$

End point values:

$f(-4) = (-4)^2 - 1 = 16 - 1 = 15$ and

$f(1) = (1)^2 - 1 = 1 - 1 = 0$

Thus $M = 15$ at $x = -4$ and $m = -1$ at $x = 0$.

10. $f(x) = 4 - x^3, \quad x \in [-2, 1]$

$f'(x) = -3x^2$

For critical points, $f'(x) = 0$

$-3x^2 = 0 \Rightarrow x = 0$

We check the functional values at $x = 0$ and at the end points $x = -2$ and $x = 1$.

Critical point value: $f(0) = 4 - (0)^3 = 4$

End point values:

$f(-2) = 4 - (-2)^3 = 4 + 8 = 12$

and $f(1) = 4 - (1)^3 = 4 - 1 = 3$

Thus $M = 12$ at $x = -2$ and $m = 3$ at $x = 1$

11. $f(x)=-3x^{\frac{2}{3}}, \quad x\in[-1,1]$

$$f'(x)=-3\left(\frac{2}{3}\right)x^{\frac{2}{3}-1}=-2x^{-\frac{1}{3}}=-\frac{2}{x^{\frac{1}{3}}}$$

Since $f'(x)$ is undefined for $x=0$, so critical point is $x=0$.

We check the functional values at $x=0$ and at the end points $x=-1$ and $x=1$.

Critical point value: $f(0)=-3(0)^{\frac{2}{3}}=0$

End point value: $f(-1)=-3(-1)^{\frac{2}{3}}=-3$

and $f(1)=-3(1)^{\frac{2}{3}}=-3$

Thus $M=0$ at $x=0$ and $m=-3$ at $x=-1,1$

12. $f(x)=x^{\frac{5}{3}} \quad , \quad x\in[-1,8]$

$$f(x)=x^{\frac{5}{3}-1}=\frac{5}{3}x^{\frac{2}{3}}$$

For critical points, $f'(x)=0$

$$\frac{5}{3}x^{\frac{2}{3}}=0\Rightarrow x=0$$

We check the function values at $x=0$ and at the end points $x=-1$ and $x=8$.

Critical point value: $f(0)=\frac{5}{3}(0)^{\frac{2}{3}}=0$

End point values: $f(-1)=\frac{5}{3}(-1)^{\frac{2}{3}}=\frac{5}{3}$

and $f(8)=\frac{5}{3}(8)^{\frac{2}{3}}=\frac{5}{3}(4)=\frac{20}{3}$

Thus $M=\frac{20}{3}$ at $x=8$ and $m=0$ at $x=0$

In Exercise $(13-20)$, find the local (relative) maximum and minimum values of each function on the given interval.

13. $f(x)=x^2-x-2$

$f'(x)=2x-1$

$f''(x)=2$ .....(1)

For critical points, $f'(x)=0$

$2x-1=0\Rightarrow x=\frac{1}{2}$

Put $x=\frac{1}{2}$ in (1)

$f''\left(\frac{1}{2}\right)=2>0$

$\Rightarrow f$ has relative minimum at $x=\frac{1}{2}$.

Min Value

$$f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2-\frac{1}{2}-2=\frac{1}{4}-\frac{1}{2}-2$$

$$f\left(\frac{1}{2}\right)=\frac{1-2-8}{4}=-\frac{9}{4}$$

14. $f(x)=5x^2-6x+2$

$f'(x)=10x-6$

$f''(x)=10$ .....(1)

For critical points, $f'(x)=0$

$10x-6=0\Rightarrow x=\frac{6}{10}\Rightarrow x=\frac{3}{5}$

Put $x=\frac{3}{5}$ in (1)

$f''\left(\frac{3}{5}\right)=10>0$

$\Rightarrow f$ has relative minimum at $x=\frac{3}{5}$.

Min value

$$f\left(\frac{3}{5}\right)=5\left(\frac{3}{5}\right)^2-6\left(\frac{3}{5}\right)+2$$

$$=5\left(\frac{9}{25}\right)-\frac{18}{5}+2$$

$$=\frac{9}{5}-\frac{18}{5}+2=\frac{9-18+10}{5}=\frac{1}{5}$$

15. $f(x)=x^3$

$f'(x)=3x^2$ .....(1)

For critical points, $f'(x)=0$

$3x^2=0 \Rightarrow x=0$

Before $x=0$

Put $x=-0.1$ in (1)

$\Rightarrow f'(-0.1)=3(-0.1)^2>0$

After $x=0$

Put $x=0.1$ in (1)

$f'(0.1)=3(0.1)^2>0$

Since $f'(x)>0$ before and after $x=0$, is the point of inflection:

Now, $f(0)=(0)^3=0$

Hence $(0,0)$ is the value at point of inflection,

16. $f(x)=5+3x-x^3$.

$f'(x)=3-3x^2$

$f''(x)=-6x$ .....(1)

For critical points, $f'(x)=0$

$3-3x^2=0$

$3x^2=3 \Rightarrow x^2=1 \Rightarrow x=\pm 1$

The critical points are -1 and 1.

Put $x=-1$ in (1)

$\Rightarrow f''(-1)=-6(-1)=6>0$

$\Rightarrow f$ has relative minima at $x=-1$.

Min value

$f(-1)=5+3(-1)-(-1)^3=5-3+1=3$

Again,

Put $x=1$ in (1)

$f''(1)=-6(1)=-6<0$

$\Rightarrow f$ has relative maxima at $x=1$

Max value

$f(1)=5+3(1)-(1)^3=5+3-1=7$

17. $f(x)=(x-1)(x-2)^2$

$=(x-1)(x^2-4x+4)$

$=x^3-4x^2+4x-x^2+4x-4$

$f(x)=x^3-5x^2+8x-4$

$f'(x)=3x^2-10x+8$

$f''(x)=6x-10$ .....(1)

For critical points, $f'(x)=0$

$3x^2-10x+8=0$

$3x^2-6x-4x+8=0$

$3x(x-2)-4(x-2)=0$

$(x-2)(3x-4)=0$

$x-2=0 \quad , \quad 3x-4=0$

$x=2 \quad , \quad x=\frac{4}{3}$

The critical points are 2 and $\frac{4}{3}$.

Put $x=2$ in (1)

$\Rightarrow f''(2)=6(2)-10=12-10=2>0$

$\Rightarrow f$ has relative minima at $x=2$.

Min value

$f(2)=(2-1)(2-2)^2=0$

Again,

Put $x=\frac{4}{3}$ in (1)

$f''\left(\frac{4}{3}\right)=6\left(\frac{4}{3}\right)-10=8-10=-2<0$

$\Rightarrow f$ has relative maxima at $x=\frac{4}{3}$.

Max value

$f\left(\frac{4}{3}\right)=\left(\frac{4}{3}-1\right)\left(\frac{4}{3}-2\right)^2=\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)^2$

$=\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)=\frac{4}{27}$.

18. $f(x)=3x^2$

$f'(x)=6x$

$f''(x)=6$ .....(1)

For critical points, $f'(x)=0$

$6x=0 \Rightarrow x=0$

Put $x=0$ in (1)

$\Rightarrow f''(0)=6>0$

$\Rightarrow f$ has relative minima at $x=0$.

Min value

$f(0)=3(0)^2=0$

Min value

$f(0)=3(0)^2=0$

19. $f(x)=3x^2-4x+5$

$f'(x)=6x-4$

$f''(x)=6$ .....(1)

For critical points, $f'(x)=0$

$6x-4=0$

$6x=4 \Rightarrow x=\frac{2}{3}$

Put $x=\frac{2}{3}$ in (1)

$\Rightarrow f''\left(\frac{2}{3}\right) \quad 6>0$

$\Rightarrow f$ has relative minima at $x=\frac{2}{3}$

Min value

$f\left(\frac{2}{3}\right)=3\left(\frac{2}{3}\right)^2-4\left(\frac{2}{3}\right)+5$

$=3\left(\frac{4}{9}\right)-\frac{8}{3}+5$

$=\frac{4}{3}-\frac{8}{3}+5=\frac{4-8+15}{3}=\frac{11}{3}$

20. $f(x)=1-x^3$

$f'(x)=-3x^2$ .....(1)

For critical points, $f'(x)=0$

$-3x^2=0 \Rightarrow x=0$

Before $x=0$

Put $x=-0.9$ in (1)

$\Rightarrow f'(-0.9)=-3(-0.9)^2<0$

After $x=0$

Put $x=0.1$ x in (1)

$f'(0.1)=-3(0.1)^2<0$

Since $f'(x)<0$ before and after $x=0$;

$x=0$ is the point of inflection.

Now

$f(0)=1-(0)^3=1$

Hence value of point of inflection is $(0,1)$

In Exercise $(21-26)$, Examine the location of all critical points and determine their nature.

21. $f(x)=3x^2-48x+100$

$f'(x)=6x-48$

$f''(x)=6$ .....(1)

For critical points, $f'(x)=0$

$6x-48=0$

$6x-48 \Rightarrow x=8$

$f(8)=3(8)^2-48(8)+100$

$=-192-384+100=-92$

So the location of critical point is $(8,-92)$

Put $x=8$ in (1)

$f''(8)=6>0$

$\Rightarrow f$ has minimum value at $x=8$

22. $f(x)=x^2-8x+4$

$f'(x)=2x-8$

$f''(x)=2$ .....(1)

For critical points, $f'(x)=0$

$2x-8=0 \Rightarrow x=4$

$f(4) = (4)^2 - 8(4) + 4$

$= 16 - 32 + 4 = -12$

So the location of critical point is $(4,-12)$

Put $x = 4$ in (1)

$f''(4) = 2 > 0$

$\Rightarrow f$ has minimum value at $x = 4$

23.

$f(x) = -10x^3 + 5$

$f'(x) = -30x^2$ .....(1)

For critical points, $f'(x) = 0$

$-3x^2 = 0 \Rightarrow x = 0$

$f(0) = -10(0)^3 + 5 = 5$

So the location of critical point is $(0,5)$

Before $x = 0$

Put $x = -0.1$ in (1)

$f'(-0.1) = -30(-0.1)^2 < 0$

After $x = 0$

Put $x = 0.1$ in (1)

$f'(0.1) = -30(0.1)^2 < 0$

$\Rightarrow f'(x) < 0$ before and after $x = 0$, so $(0,5)$ is the point of inflection.

24. $f(x) = -5x^2 - 10$

$f'(x) = -25x^4$ .....(1)

For critical points, $f'(x) = 0$

$-25x^4 = 0 \Rightarrow x = 0$

$f(0) = -5(0)^5 - 10 = -10$

So the location of critical point is $(0,-10)$

Before $x = 0$

Put $x = -0.1$ in (1)

$\Rightarrow f'(-0.1) = -25(-0.1)^4 < 0$

After $x = 0$

Put $x = 0.1$ in (1)

$f'(0.1) = -25(0.1)^4 < 0$

$\Rightarrow f'(x) < 0$ before and after $x = 0$, so $(0,-10)$ is the point of inflection.

25. $f(x) = \frac{4x^2}{15} + 4$

$f'(x) = \frac{8x}{15}$

$f''(x) = \frac{8}{15}$ .....(1)

For critical points, $f'(x) = 0$

$\frac{8x}{15} = 0 \Rightarrow x = 0$

So the location of critical point is $(0,4)$.

$f(0) = \frac{4}{15}(0)^2 + 4 = 4$

Put $x = 0$ in (1)

$\Rightarrow f''(0) = \frac{8}{15} > 0$

$\Rightarrow f$ has minimum value at $x = 0$.

26. $f(x) = (x+10)^4$

$f'(x) = 4(x+10)^3$

$f''(x) = 12(x+10)^2$ .....(1)

For critical points, $f'(x) = 0$

$4(x+10)^3 = 0$

$x + 10 = 0 \Rightarrow x = -10$

$f(-10) = (-10+10)^4 = 0$

So the location of critical point is $(-10,0)$

Before $x = -10$

Put $x = -10.1$ in (1)

$\Rightarrow f'(-10.1) = 4(-10.1+10)^3 < 0$

After $x = -10$

Put $x = -9.9$ in (1)

$f'(-9.9) = 4(-9.9+10)^3 > 0$

We see that $f'(x) < 0$ before $x = -10$ and $f'(x) > 0$ after $x = -10$, so f has Minimum value at $x = -10$.

Exercises 5.3

1. Find out the production units x for which cost is minimum if cost function $C(x)$ is given by:

$C(x) = x^3 - 120x^2 + 4500x$

Solution:

$$C(x) = x^3 - 120x^2 + 4500x$$

$$\Rightarrow \frac{dC}{dx} = 3x^2 - 240x = 4500$$

$$\Rightarrow \frac{d^2C}{dx^2} = 6x - 240 \text{ ----------(1)}$$

For critical points, $\frac{dC}{dx} = 0$

$$3x^2 - 240x + 4500 = 0$$

$$\Rightarrow x^2 - 80x + 1500 = 0$$

$$x^2 - 50x - 30x + 1500 = 0$$

$$x(x-50) - 30(x-50) = 0$$

$$(x-30)(x-50) = 0$$

$$x - 30 = 0 \quad , \quad x - 50 = 0$$

$$x = 30 \quad , \quad x = 50$$

Put $x = 30$ in (1)

$$\frac{d^2C}{dx^2} = 6(30) - 240 = 180 - 240 = -60 < 0$$

We are not interested in max value

Put $x = 50$ in (1)

$$\frac{d^2C}{dx^2} = 6(50) - 240 = 300 - 240 = 60 > 0$$

$\Rightarrow$ cost is minimum at $x = 50$

Min value

$$C(50) = (50)^3 - 120(50)^2 + 4500(50)$$

$$= 50,000$$

Hence cost will be minimum $Rs.50,000$ when 50 units are produced.

2. Find the production unit x for which cost is minimum if cost function $C(x)$ is given by: $C(x) = \frac{x^3}{3} - 400x^2 + 1,50,00$

Solution:

$$C(x) = \frac{x^3}{3} - 400x^2 + 1,50,00$$

$$\Rightarrow \frac{dC}{dx} = x^2 - 800x$$

$$\Rightarrow \frac{d^2C}{dx^2} = 2x - 800$$

For critical points, $\frac{dC}{dx} = 0$

$$x^2 - 800x = 0$$

$$x(x - 800) = 0$$

$$x = 0 \quad , \quad x - 800 = 0$$

$$x = 800$$

Put $x = 800$ in (1)

$$\frac{d^2C}{dx^2} = 2(800) - 800 = 800 > 0$$

$\Rightarrow$ cost is minimum when $x = 800$ units are produced.

3. Find out the output x that maximizes profit P, if P is defined as:

$P(x) = -2x^2 + 600x + 150$

Solution:

$P(x) = -2x^2 + 600x + 150$

$\Rightarrow \frac{dP}{dx} = -4x + 600$

$\Rightarrow \frac{d^2P}{dx^2} = -4 \ldots\ldots\ldots\ldots(1)$

For critical points, $\frac{dP}{dx} = 0$

$-4x + 600 = 0$

$-4x = -600$

$x = 150$

Put $x = 150$ in (1)

$\frac{d^2P}{dx^2} = -4 < 0$

$\Rightarrow P$ will be maximum when the output $x = 150$ units.

4. The profit P at different levels of output x of Abdullah Ltd is given by

$P(x) = 3000 + 1200x - 2x^2$

(i)What is the output at which the profit will be maximum.

(ii)What will be the maximum amount of profit?

Solution:

(i)

$P(x) = 3000 + 1200x - 2x^2$

$\Rightarrow \frac{dP}{dx} = 1200 - 4x$

$\Rightarrow \frac{d^2P}{dx^2} = -4 \ldots\ldots\ldots\ldots(1)$

For critical points, $\frac{dP}{dx} = 0$

$1200 - 4x = 0$

$4x = 1200$

$x = 300$

Put $x = 300$ in (1)

$\frac{d^2P}{dx^2} = -4 < 0$

$\Rightarrow P$ will be maximum at $x = 300$

The profit will be maximum when output is 300 units.

(ii) Max Profit

$P(300) = 300 + 1200(300) - 2(300)^2$

$= 3000 + 360000 - 180000$

$= 183,000$

Hence Profit will be $Rs.1,83,000$.

5. Socks Knitting company inter-loop sells a special brand of socks at $1,000 per dozen. The cost function of making x dozen in one day is given by

$C(x) = 6 + 2x + 0.01x^2$

How many dozen socks should be produced and sold to earn the maximum profit?

Solution:

$C(x) = 6 + 2x + 0.01x^2$

$R(x) = 1000x$ ($R(x)$: Revenue)

$P(x) = R(x) - C(x)$

$= 1000x - 6 - 2x - 0.01x^2$

$P(x) = 998x - 6 - 0.01x^2$

$\Rightarrow \frac{dP}{dx} = 998 - 0.02x$

$\Rightarrow \frac{d^2P}{dx^2} = -0.02 \ldots\ldots\ldots(1)$

For critical points, $\frac{dP}{dx} = 0$

$998 - 0.02x = 0$

$0.02x = 998$

$x = \frac{998}{0.02} \Rightarrow x = 44900$

Put $x = 44900$ in (1)

$\frac{d^2P}{dx^2} = -0.02 < 0$

$\Rightarrow p$ will be maximum at $x = 44900$

Hence inter-loop can get maximum profit when 44900 dozen socks per day will be

produced.

6. The perimeter of a triangle is $16cm$, what are length of the other sides for maximum area of the triangle?

Solution:

One side of triangle $=6cm$

Let the second side of triangle $=xcm$

The third side of triangle $=16-(6+x)$

$=16-6-x=10-x$

Let $f(x)$ be the square of the area of triangle, then

$f(x)=8(8-6)(8-x)(8-10+x)$

using

$\Delta=\sqrt{s(s-a)(s-b)(s-c)},\ s=\frac{16}{2}=8$

$f(x)=8(2)(8-x)(x-2)$

$=16(8x-16-x^2+2x)$

$f(x)=16(-x^2+10x-16)$

$f'(x)=16(-2x+10)$

$f''(x)=16(-2)=-32$ .....(1)

For critical points, $f'(x)=0$

$16(-2x+10)=0$

$-2x+10=0\Rightarrow -2x=-10\Rightarrow x=5$

Put $x=5$ in (1)

$f''(5)=-32<0$

$\Rightarrow f$ has maximum value at $x=5$ so the other sides having maximum area are 5,5 and 6.

7. Find the dimentions of a rectangle of largest area having perimeter $120cm$.

Solution:

Let length of rectangle $=x$

Breadth of rectangle $=y$

Perimeter of rectangle $=120cm$

$2(x+y)=120$

$x+y=60$

$y=60-x$ .....(1)

Let A be the area of rectangle, then

$A=xy$

$A=x(60-x)$ using (1)

$A=60x-x^2$

Diff .. w.r.t x

$\frac{dA}{dx}=60-2x$

$\Rightarrow \frac{d^2A}{dx^2}=-2$ .....(2)

For critical point, $\frac{dA}{dx}=0$

$60-2x=0\Rightarrow 2x=60\Rightarrow x=30$

Put $x=30$ in (2)

$\frac{d^2A}{dx^2}=-2<0$

$\Rightarrow A$ has maximum value at $x=30$

Hence the dimension for maximum area are

Length $=x=30cm$

Breadth $=y=60-30=30cm$

8. A box with a square base and open top is to have a volume of $4dm^3$. Find the dimension of the box which will require the least material.

Solution:

Let x be the length of each side of square base and y be the length of 3rd side.

Since volume of box is $4dm^3$, so

$x^2y=4\Rightarrow y=\frac{4}{x^2}$ .....(1)

Let S be the surface of box, then

$S=x^2+4xy$

$S=x^2+4x\left(\frac{4}{x^2}\right)$ by (1)

$S=x^2+\frac{16}{x}$

Diff w.r.t x

$$\frac{ds}{dx}=2x-\frac{16}{x^2}$$

$$\frac{d^2s}{dx^2}=2+\frac{32}{x^3}\ldots\ldots(2)$$

For extreme value, put $\frac{ds}{dx}=0$

$$2x-\frac{16}{x^2}=0$$

$$2x^3-16=0\Rightarrow x^3=8\Rightarrow x=2$$

Put $x=2$ in (2)

$$\frac{d^2s}{dx^2}=2+\frac{32}{(2)^3}=6>0$$

So the surface area will be minimum at $x=2$

Put $x=2$ in (1)

$$y=\frac{4}{(2)^2}=1$$

So the box will require least material for dimension $2\times2\times1$

9. Find the lengths of the sides of a variable rectangle having area $36cm^2$, when its perimeter is minimum.

Solution:

Let length of rectangle $=xcm$

Breadth of rectangle $=ycm$

Area $=36cm^2$

$$xy=36\Rightarrow y=\frac{36}{x}\ldots\ldots\ldots(1)$$

Now, Perimeter $=2(x+y)$

$$P=2\left(x+\frac{36}{x}\right)\quad\text{using (1)}$$

$$P=2x+\frac{72}{x}$$

$$\Rightarrow\frac{dP}{dx}=2-\frac{72}{x^2}$$

$$\Rightarrow\frac{d^2P}{dx^2}=\frac{144}{x^3}\text{----------(2)}$$

For critical points, $\frac{dP}{dx}=0$

$$2-\frac{72}{x^2}=0\Rightarrow2=\frac{72}{x^2}$$

$$x^2=36\Rightarrow x=\pm6$$

Put $x=6$ in (2) ($x=-6$ is not possible)

$$\frac{d^2P}{dx^2}=\frac{144}{(6)^3}>0$$

$\Rightarrow$ Perimeter will be minimum whe $x=6$

Hence

Length $=x=6cm$

Breadth $=y=\frac{36}{x}=\frac{36}{6}=6cm$

10. Find the dimensions of rectangular garden having perimeter 80 meters if its area is to be maximum.

Solution:

Let length of rectangle $=xm$

Breadth of rectangle $=ym$

Perimeter $=80m$

$$2(x+y)=80\Rightarrow x+y=40$$

$$y=40-x\ldots\ldots(1)$$

Let A be the area of rectangle garden, then

$$A=xy$$

$$A=x(40-x)$$

$$A=40x-x^2$$

$$\Rightarrow\frac{dA}{dx}=40-2x$$

$$\Rightarrow\frac{d^2A}{dx^2}=-2\ldots\ldots\ldots(2)$$

For critical points, $\frac{dA}{dx}=0$

$$40-2x=0$$

$$\Rightarrow2x=40\Rightarrow x=20$$

Put $x=20$ in (2)

$\frac{d^2A}{dx^2} = -2 < 0$

$\Rightarrow$ Area will be maximum when $x = 20$

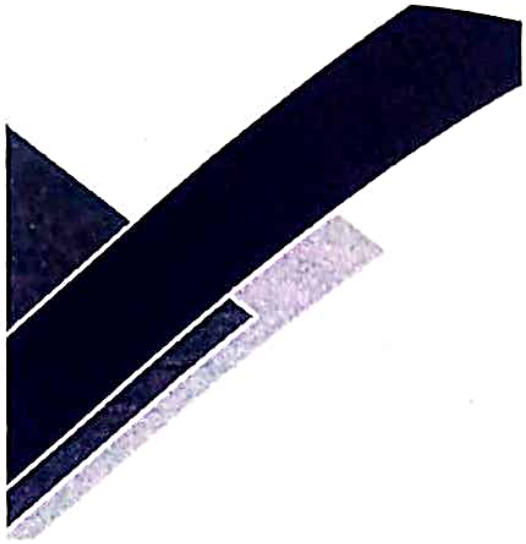
Hence

Length of rectangle garden $= x = 20m$

Breadth of rectangle garden $= y = 40 - x$

$= 40 - 20 = 20m \quad y = 40 - 20$

CHAPTER 6
INTEGRATION AND INDEFINITE INTEGRALS



Exercises 6.1

In exercises $(1-24)$, find the indefinite integral.

Solutions:

1. $\int 20\,dx = 20x + c$

2. $\int -4x\,dx = -4\dfrac{x^2}{2} + c = -2x^2 + c$

3. $\int \dfrac{-3x}{4}\,dx = -\dfrac{3x^2}{8} + c$

4. $\int (2x+5)\,dx$

$= \dfrac{2x^2}{2} + 5x + c$

$= x^2 + 5x + c$

5. $\int (10-4x)\,dx$

$= 10x - \dfrac{4x^2}{2} + c = 10x - 2x^2 + c$

6. $\int (3x^2 - 2x + 4)\,dx$

$= \dfrac{3x^3}{3} - \dfrac{2x^2}{2} + 4x + c$

$= x^3 - x^2 + 4x + c$

7. $\int \left(\dfrac{x}{2} + \dfrac{1}{4}\right)dx = \dfrac{x^2}{4} + \dfrac{x}{4} + c$

8. $\int (-6x^3 + 2x + 1)\,dx$

$= -6\dfrac{x^4}{4} + \dfrac{2x^2}{2} + x + c$

$= \dfrac{-3}{2}x^4 + x^2 + x + c$

9. $\int (-3x^2 + 5)\,dx$

$= -\dfrac{3x^3}{3} + 5x + c$

$= -x^3 + 5x + c$

10. $\int (4x^3 + 2x^2 + 3)\,dx$

$= \dfrac{4x^4}{4} + \dfrac{2x^3}{3} + 3x + c$

$= x^4 + \dfrac{2}{3}x^3 + 3x + c$

11. $\int (8x^3 + 3x - 6)\,dx$

$= \dfrac{8x^4}{4} + \dfrac{3x^2}{2} - 6x + c$

$$= 2x^4 + \frac{3x^2}{2} - 6x + c$$

12. $\int \sqrt[5]{x}\,dx$

$$= \int (x)^{\frac{1}{5}}\,dx$$

$$= \frac{x^{\frac{1}{5}+1}}{\frac{1}{5}+1} + c$$

$$= \frac{x^{\frac{6}{5}}}{\frac{6}{5}} + c$$

$$= \frac{5}{6}x^{\frac{6}{5}} + c$$

13. $\int 2\sqrt[3]{x}$

$$= 2\int (x)^{\frac{1}{3}}\,dx$$

$$= 2\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{3}{2}x^{\frac{4}{3}} + c$$

14. $\int (ax^4 + bx^2)\,dx = \frac{ax^5}{5} + \frac{bx^3}{3} + c$

15. $\int \frac{1}{x^n}\,dx$

$$= \int x^{-n}\,dx$$

$$= \frac{x^{-n+1}}{-n+1} + c$$

$$= \frac{x^{1-n}}{1-n} + c$$

16. $\int \frac{16}{x^2}\,dx$

$$= 16\int x^{-2}\,dx$$

$$= 16\frac{x^{-1}}{-1} + c = -\frac{16}{x} + c$$

17. $\int (ax^4 + bx^3 + cx^2 + dx + e)\,dx$

$$= \frac{ax^5}{5} + \frac{bx^4}{4} + \frac{cx^3}{3} + \frac{dx^2}{2} + ex + c'$$

18. $\int \frac{-2}{\sqrt{x}}dx$

$$= -2\int (x)^{-\frac{1}{2}}\,dx$$

$$= -2\frac{(x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -4\sqrt{x} + c$$

19. $\int (4ax^3 + 3bx^2 + 2cx)\,dx$

$$= \frac{4ax^4}{4} + \frac{3bx^3}{3} + \frac{2cx^2}{2} + c'$$

$$= ax^4 + bx^3 + cx^2 + c'$$

20. $\int \frac{-15}{\sqrt[7]{x}}dx$

$$= -15\int \frac{1}{(x)^{\frac{1}{7}}}dx$$

$$= -15\int (x)^{\frac{-1}{7}}\,dx$$

$$= -15\frac{(x)^{\frac{-1}{7}+1}}{-\frac{1}{7}+1} + c$$

$$= -15\frac{(x)^{\frac{6}{7}}}{\frac{6}{7}} + c$$

$$= -\frac{35}{2}(x)^{\frac{6}{7}} + c$$

21. $\int \left(\frac{1}{x^2} + \frac{3}{x^3}\right)dx$

$$= \int \frac{1}{x^2}dx + \int \frac{3}{x^3}dx$$

$$= \int x^{-2}dx + 3\int x^{-3}\,dx$$

$$= \frac{x^{-1}}{-1} + 3\cdot\frac{x^{-2}}{-2} + c$$

$$= -\frac{1}{x} - \frac{3}{2x^2} + c$$

22. $\int \frac{4\sqrt[4]{x}}{3}dx$

$$= \frac{4}{3}\int (x)^{\frac{1}{4}}\,dx$$

$$= \frac{4}{3}\frac{(x)^{\frac{1}{4}+1}}{\frac{1}{4}+1} + c$$

$$= \frac{4}{3}\frac{(x)^{\frac{5}{4}}}{\frac{5}{4}} + c$$

$$= \frac{16}{15}(x)^{\frac{5}{4}} + c$$

23. $\int (2x-5)(3x+1)\,dx$

$$= \int \left(6x^2 + 2x - 15x - 5\right)dx$$

$$= \int \left(6x^2 - 13x - 5\right)dx$$

$$= \frac{6x^3}{3} - \frac{13x^2}{2} - 5x + c$$

$$= 2x^3 - \frac{13x^2}{2} - 5x + c$$

24. $\int \frac{\left(t^2+3\right)^2}{t^6}dt$

$$= \int \frac{t^4 + 6t^2 + 9}{t^6}\,dt$$

$$= \int \left(\frac{t^4}{t^6} + \frac{6t^2}{t^6} + \frac{9}{t^6}\right)dt$$

$$= \int \left(\frac{1}{t^2} + \frac{6}{t^4} + \frac{9}{t^6}\right)dt$$

$$= \int \left(t^{-2} + 6t^{-4} + 9t^{-6}\right)dt$$

$$= \frac{t^{-1}}{-1} + \frac{6t^{-3}}{-3} + \frac{9t^{-5}}{-5} + c$$

$$= -\frac{1}{t} - \frac{2}{t^3} - \frac{9}{5t^5} + c$$

Exercises 6.2

In exercises $(1-30)$, find the indefinite integral (if possible).

1. $\int (x+10)^5\,dx$

Here $f(x) = x+10$, $f'(x) = 1$, then

$$= \frac{(x+10)^6}{6} + c$$

2. $\int \sqrt{x-10}\,dx = \int (x-10)^{\frac{1}{2}}\,dx$

Here $f(x) = x-10$, $f'(x) = 1$, then

$$= \frac{(x-10)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{3}(x-10)^{\frac{3}{2}} + c$$

3. $\int 4(2x+5)^3\,dx$

$$\int 4(2x+5)^3\,dx = 2\int (2x+5)^3\,(2)\,dx$$

Here,

$f(x) = (2x+5)$, $f'(x) = 2$, then

$$= 2\frac{(2x+5)^4}{4} + c$$

$$= \frac{(2x+5)^4}{2} + c$$

4. $\int \left(\frac{x}{3}+15\right)^4 \left(\frac{1}{3}\right) dx$

Here $f(x) = \frac{x}{3}+15$, $f'(x) = \frac{1}{3}$, then

$$= \frac{\left(\frac{x}{3}+15\right)^5}{5} + c$$

$$= \frac{1}{5}\left(\frac{x}{3}+15\right)^5 + c$$

5. $\int (x^2+3)^4 (2x)\, dx$

Here $f(x) = x^2+3$, $f'(x) = 2x$, then

$$= \frac{(x^2+3)^5}{5} + c$$

6. $\int (x^3+1)^3 (x^2)\, dx$

$$= \frac{1}{3}\int (x^3+1)^3 (3x^2)\, dx$$

Here $f(x) = x^3+1$, $f'(x) = 3x^2$, then

$$= \frac{1}{3}\frac{(x^3+1)^4}{4} + c$$

$$= \frac{(x^3+1)^4}{12} + c$$

7. $\int (x^2+5)^4 (x)\, dx$

$$= \frac{1}{2}\int (x^2+5)^4 (2x)\, dx$$

Here $f(x) = x^2+5$, $f'(x) = 2x$, then

$$= \frac{1}{2}\frac{(x^2+5)^5}{5} + c \quad = \frac{(x^2+5)^5}{10} + c$$

8. $\int (2x^2-4x)^8 (x-1)\, dx$

$$= \frac{1}{4}\int (2x^2-4x)^8 (4x-4)\, dx$$

Here $f(x) = 2x^2-4x$, $f'(x) = 4x-4$, then

$$= \frac{1}{4}\frac{(2x^2-4x)^9}{9} + c$$

$$= \frac{(2x^2-4x)^9}{36} + c$$

9. $\int \left(\frac{x^2}{4}-\frac{x}{2}\right)^5 (x-1)\, dx$

$$= 2\int \left(\frac{x^2}{4}-\frac{x}{2}\right)\left(\frac{x}{2}-\frac{1}{2}\right) dx$$

Here $f(x) = \frac{x^2}{4}-\frac{x}{2}$, $f'(x) = \frac{x}{2}-\frac{1}{2}$, then

$$= 2\frac{\left(\frac{x^2}{4}-\frac{x}{2}\right)^6}{6} + c$$

$$= \frac{1}{3}\left(\frac{x^2}{4}-\frac{x}{2}\right)^6 + c$$

10. $\int \sqrt{x^2+8}\ (2x)\, dx$

$$= \int (x^2+8)^{\frac{1}{2}} (2x)\, dx$$

Here $f(x) = x^2+8$, $f'(x) = 2x$, then

$$= \frac{(x^2+8)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{3}(x^2+8)^{\frac{3}{2}} + c$$

11. $\int \frac{2x}{\sqrt{x^2+8}}\, dx$

$$=\int (x^2+8)^{-\frac{1}{2}}(2x)dx$$

Here $f(x)=x^2+8$, $f'(x)=2x$, then

$$=\frac{(x^2+8)^{\frac{1}{2}}}{\frac{1}{2}}+c$$

$$=2\sqrt{x^2+8}+c$$

12. $\int \frac{3x^2}{(x^3+4)^3}dx$

$$=\int (x^3+4)^{-3}(3x^2)dx$$

Here $f(x)=x^3+4$, $f'(x)=3x^2$, then

$$=\frac{(x^3+4)^{-2}}{-2}+c$$

$$=-\frac{1}{2(x^3+4)^2}+c$$

13. $\int (4x^3+8x^2)^5(3x^2+4x)dx$

$$=\frac{1}{4}\int (4x^3+8x^2)^5(12x^2+16x)dx$$

Here

$f(x)=4x^3+8x^2$, $f'(x)=12x^2+16x$, then

$$=\frac{1}{4}\frac{(4x^3+8x^2)^6}{6}+c$$

$$=\frac{(4x^3+8x^2)^6}{24}+c$$

14. $\int \sqrt{9x-3x^2}(3-2x)dx$

$$=\frac{1}{3}\int (9x-3x^2)^{\frac{1}{2}}(9-6x)dx$$

Here

$f(x)=9x-3x^2$, $f'(x)=19-6x$, then

$$=\frac{1}{3}\frac{(9x-3x^2)^{\frac{3}{2}}}{\frac{3}{2}}+c$$

$$=\frac{2}{9}(9x-3x^2)^{\frac{3}{2}}+c$$

15. $\int e^{x-8}dx$

Here $f(x)=x-8$, $f'(x)=1$

{Using $\int e^{f(x)}f'(x)dx=e^{f(x)}+c$}

$$=e^{x-8}+c$$

16. $\int e^{3x}dx$

$$=\frac{1}{3}\int e^{3x}(3)dx$$

Here $f(x)=3x$, $f'(x)=3$

{Using $\int e^{f(x)}f'(x)dx=e^{f(x)}+c$}

$$=\frac{1}{3}e^{3x}+c$$

17. $\int e^{x^2+4x}(x+2)dx$

$$=\frac{1}{2}\int e^{x^2+4x}(2x+4)dx$$

Here $f(x)=x^2+4x$, $f'(x)=2x+4$

{Using $\int e^{f(x)}f'(x)dx=e^{f(x)}+c$}

$$=\frac{1}{2}e^{x^2+4x}+c$$

18. $\int \frac{-x}{x^2+5}dx$

$$=-\frac{1}{2}\int \frac{2x}{x^2+5}dx$$

Here $f(x)=x^2+5$, $f'(x)=2x$

{Using $\int \frac{f'(x)}{f(x)}dx=\ln f(x)+c$}

$$=-\frac{1}{2}\ln(x^2+5)+c$$

19. $\int \frac{8x}{100+x^2}\,dx$

$= 4\int \frac{2x}{100+x^2}\,dx$

Here $f(x) = 100 + x^2$, $f'(x) = 2x$

{Using $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$ }

$= 4\ln(100 + x^2) + c$

20. $\int \frac{18}{6x+5}\,dx$

$= 3\int \frac{6}{6x+5}\,dx$

Here $f(x) = 6x + 5$, $f'(x) = 6$

{Using $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$ }

$= 3\ln(6x+5) + c$

21. $\int \frac{dx}{ax+b}$

$= \frac{1}{a}\int \frac{a}{ax+b}\,dx$

Here $f(x) = ax + b$, $f'(x) = a$

{Using $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$ }

$= \frac{1}{a}\ln(ax+b) + c$

22. $\int \frac{ax-1}{ax^2-2x}\,dx$

$= \frac{1}{2}\int \frac{2ax-2}{ax^2-2x}\,dx$

Here $f(x) = ax^2 - 2x$, $f'(x) = 2ax - 2$

{Using $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$ }

$= \frac{1}{2}\ln(ax^2 - 2x) + c$

23. $\int \frac{2ax+b}{ax^2+bx}\,dx$

Here

$f(x) = ax^2 + bx$, $f'(x) = 2ax + b$

{Using $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$ }

$= \ln(ax^2 + bx) + c$

24. $\int \frac{a-bx}{2ax-bx^2}\,dx$

$= \frac{1}{2}\int \frac{2a-2bx}{2ax-bx^2}\,dx$

Here

$f(x) = 2ax - bx^2$, $f'(x) = 2a - 2bx$

{Using $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$ }

$= \frac{1}{2}\ln(2ax - bx^2) + c$

25. $\int (\sin x \cos x)\,dx$

Here $f(x) = \sin x, f'(x) = \cos x$, then

$= \frac{(\sin x)^2}{2} + c$

$= \frac{\sin^2 x}{2} + c$

26. $\int \frac{\tan x}{\cos x}\,dx$

$= \int \sec x \tan x\,dx \left(\because \frac{1}{\cos x} = \sec x\right)$

$= \sec x + c$

27. $\int (1+\tan x)^2 . \sec^2 x\,dx$

Here $f(x) = 1 + \tan x$, $f'(x) = \sec^2 x$,

then

$= \frac{(1+\tan x)^3}{3} + c$

28. $\int \sec^2 3x\, dx = \dfrac{\tan 3x}{3} + c$

29. $\int \operatorname{cosec} 2x \cot 2x\, dx$

$= -\dfrac{\operatorname{cosec} 2x}{2} + c$

30. $\int \tan^2 x\, dx$

$= \int (\sec^2 x - 1)\, dx$

$= \tan x - x + c$

Exercises 6.3

In exercises $(1-18)$, evaluate the following integrals with suitable substitution (if needed).

1. $\int \dfrac{2x}{\sqrt{x^2+1}}\, dx$

$= \int (x^2+1)^{\frac{-1}{2}} (2x)\, dx$

Let $x^2+1 = t$, then $2x\, dx = dt$

Substituting, we have given integral.

$= \int (t)^{\frac{-1}{2}}\, dt = \dfrac{t^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2\sqrt{t} + c = 2\sqrt{x^2+1} + c$

2. $\int \dfrac{e^x}{e^x+3}\, dx$

Let $e^x + 3 = t$, then $e^x dx = dt$

Substituting, we have given integral.

$= \int \dfrac{dt}{t} = \ln t + c$

Using $\int \dfrac{f'(x)}{f(x)}\, dx = \ln f(x) + c$

$= \ln(e^x + 3) + c$

3. $\int \dfrac{\sec^2 x}{\sqrt{\tan x}}\, dx$

$= \int (\tan x)^{-\frac{1}{2}} (\sec^2 x)\, dx$

Let $\tan x = t$, then $\sec^2 x dx = dt$

Substituting, we have given integral.

$= \int (t)^{-\frac{1}{2}} dt = \dfrac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{t} + c$

$= 2\sqrt{\tan x} + c$

4. $\int (ax^2 + 2bx + c)^3 (ax+b)\, dx$

Let $ax^2 + 2bx + c = t$

then $(2ax + 2b)dx = dt$

$2(ax+b)dx = dt$

$(ax+b)dx = \dfrac{dt}{2}$

Substituting, we have given integral.

$= \int t^3 \cdot \dfrac{dt}{2} = \dfrac{1}{2} \cdot \dfrac{t^4}{4} + c$

$= \dfrac{t^4}{8} + c = \dfrac{(ax^2+2bx+c)^4}{8} + c'$

5. $\int \dfrac{2x+1}{(x^2+x+1)^{\frac{5}{2}}}\, dx$

$= \int (x^2+x+1)^{\frac{-5}{2}} (2x+1)\, dx$

Let $x^2 + x + 1 = t$,

then $(2x+1)dx = dt$

Substituting, we have given integral.

$$=\int (t)^{-\frac{5}{2}}\,dt = \frac{t^{-\frac{5}{2}+1}}{-\frac{5}{2}+1}+c = \frac{t^{-\frac{3}{2}}}{-\frac{3}{2}}+c$$

$$=-\frac{2}{3}t^{\frac{-3}{2}}+c = -\frac{2}{3t^{\frac{3}{2}}}+c$$

$$=-\frac{2}{3\left(x^2+x+1\right)^{\frac{3}{2}}}+c$$

6. $\int \frac{dx}{x(\ln 2x)^3} = \int (\ln 2x)^{-3}\cdot\frac{1}{x}dx$

Let $\ln 2x = t$

then $\frac{1}{2x}(2)dx = dt \Rightarrow \frac{1}{x}dx = dt$

Substituting, we have given integral.

$$=\int (t)^{-3}\,dt = \frac{t^{-2}}{-2}+c = -\frac{1}{2t^2}+c$$

$$=-\frac{1}{2(\ln 2x)^2}+c$$

7. $\int e^{x^2}\,x\,dx$

Let $e^{x^2}=t$, then

$$e^{x^2}\cdot(2x)dx = dt \Rightarrow e^{x^2}\cdot x\,dx = \frac{dt}{2}$$

Substituting, we have given integral.

$$=\int \frac{dt}{2} = \frac{1}{2}t+c = \frac{e^{x^2}}{2}+c$$

8. $\int \frac{\cos\theta}{1+\sin\theta}\,d\theta$

Let $1+\sin\theta = t$, then $\cos\theta\,d\theta = dt$

Substituting, we have given integral.

$$=\int \frac{dt}{t} = \ln t + c = \ln(1+\sin\theta)+c$$

9. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Let $\sqrt{x}=t$, then

$$\frac{1}{2\sqrt{x}}dx = dt \Rightarrow \frac{1}{\sqrt{x}}dx = 2\,dt$$

Substituting, we have given integral.

$$=\int \sin t\cdot 2dt = 2\int \sin t\,dt$$

$$=-2\cos t + c = -2\cos\sqrt{x}+c$$

10. $\int \frac{\cos(\ln x)}{x}\,dx$

Let $\ln x = t$, then $\frac{1}{x}dx = dt$

Substituting, we have given integral.

$$=\int \cos t\cdot dt = \sin t + c = \sin(\ln x)+c$$

11. $\int \cos^6 x\sin^3 x\,dx$

$$=\int \cos^6 x\cdot\sin x\cdot\sin^2 x\,dx$$

$$=\int \cos^6 x\left(1-\cos^2 x\right)\sin x\,dx$$

$$=\int \left(\cos^6 x-\cos^8 x\right)\sin x\,dx$$

Let $\cos x = t$, then

$$-\sin x\,dx = dt \Rightarrow \sin x\,dx = -dt$$

$$=\int \left(t^6-t^8\right)(-dt) = \int \left(t^8-t^6\right)(dt)$$

$$=\frac{t^9}{9}-\frac{t^7}{7}+c = \frac{\cos^9 x}{9}-\frac{\cos^7 x}{7}+c$$

12. $\int \cos^2\theta\sin\theta\,d\theta$

Let $\cos\theta = t$, then

$$-\sin\theta\,d\theta = dt \Rightarrow \sin\theta\,d\theta = -dt$$

Substituting, we have given integral.

$$=\int t^2(-dt) = -\int t^2(dt)$$

$$=-\frac{t^3}{3}+c = -\frac{\cos^3\theta}{3}+c$$

13. $\int \cot^2\theta\cdot\operatorname{cosec}^2\theta\,d\theta$

Let $\cot\theta = t$, then

$$-\operatorname{cosec}^2\theta\,d\theta = dt \Rightarrow \operatorname{cosec}^2\theta\,d\theta = -dt$$

Substituting, we have given integral.

$$=\int t^2(-dt) = -\int t^2(dt)$$

$$= -\frac{t^3}{3} + c = -\frac{\cot^3\theta}{3} + c$$

14. $\int \frac{\cos x}{\sin x \cdot \ln(\sin x)}\, dx$

$$= \int \frac{1}{\ln(\sin x)} \cdot \frac{\cos x}{\sin x}\, dx$$

Let $\ln(\sin x) = t$,

then $\frac{\cos x}{\sin x}\, dx = dt$

Substituting, we have given integral.

$$= \int \frac{1}{t}\, dt = \ln|t| = \ln\left|\ln(\sin x)\right| + c$$

15. $\int \frac{\cos x(\ln \sin x)}{\sin x}\, dx$

Let $\ln(\sin x) = t$,

then $\frac{1}{\sin x} \cdot \cos x\, dx = dt$

Substituting, we have given integral.

$$= \int t\, dt = \frac{t^2}{2} + c = \frac{[\ln(\sin x)]^2}{2} + c$$

16. $\int \cos\left(\sqrt{x} - \frac{x}{2}\right) \cdot \left(\frac{1}{\sqrt{x}} - 1\right) dx$

Let $\sqrt{x} - \frac{x}{2} = t$,

then $\left(\frac{1}{2\sqrt{x}} - \frac{1}{2}\right) dx = dt$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{\sqrt{x}} - 1\right) dx = dt$$

$$\Rightarrow \left(\frac{1}{\sqrt{x}} - 1\right) dx = 2\, dt$$

Substituting, we have given integral.

$$= \int \cos t\,(2\, dt) = 2\int \cos t\,(dt) = 2\sin t + c$$

$$= 2\sin\left(\sqrt{x} - \frac{x}{2}\right) + c$$

17. $\int \sqrt{e^{2x} + e^{3x}}\, dx$

$$= \int \sqrt{e^{2x}\left(1 + e^x\right)}\, dx = \int \sqrt{1 + e^x}\, .e^x\, dx$$

Put $1 + e^x = t$, then $e^x dx = dt$

$$= \int \sqrt{t}\, dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3} t^{\frac{3}{2}} + c = \frac{2}{3}\left(1 + e^x\right)^{\frac{3}{2}} + c$$

18. $\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x} + 1}$

Put $e^x = t$, then $e^x dx = dt$

$$= \int \frac{dt}{t^2 + 1} = arc\tan t + c = arc\tan e^x + c$$

Exercises 6.4

In Exercises $(1-28)$, evaluate the following (if possible) using integration by parts.

1. $\int \underset{I}{x} \cdot \underset{II}{e^x}\, dx$

$$= xe^x - \int e^x \cdot 1\, dx$$

$$= xe^x - e^x + c \;= e^x(x-1) + c$$

2. $\int \underset{I}{x^2} \cdot \underset{II}{e^{2x}}\, dx$

$$= x^2 \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot (2x)\, dx$$

$$= \frac{x^2 e^{2x}}{2} - \int \underset{II}{e^{2x}} \cdot \underset{I}{x}\, dx$$

$$= \frac{x^2 e^{2x}}{2} - \left[x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot 1\, dx\right]$$

$$= \frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{1}{2}\int e^{2x}\,dx$$

$$= \frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{1}{2}\cdot\frac{e^{2x}}{2} + c$$

$$= \frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{e^{2x}}{4} + c$$

$$= \frac{e^{2x}}{4}\left(2x^2 - 2x + 1\right) + c$$

3. $\int \ln x\,dx = \int \underset{II}{1}\cdot \underset{I}{\ln x}\,dx$

$$= \ln x \cdot x - \int x\cdot\frac{1}{x}\,dx$$

$$= x\ln x - \int dx = x\ln x - x + c$$

4. $\int \underset{II}{x^2}\cdot \underset{I}{\ln x}\,dx$

$$= \ln x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{x}dx$$

$$= \frac{x^3\ln x}{3} - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3\ln x}{3} - \frac{1}{3}\cdot\frac{x^3}{3} + c = \frac{x^3\ln x}{3} - \frac{x^3}{9} + c$$

$$= \frac{x^3}{9}(3\ln x - 1) + c$$

5. $\int \underset{II}{x^3}\cdot \underset{I}{\ln x}\,dx$

$$= \ln x\cdot\frac{x^4}{4} - \int\frac{x^4}{4}\cdot\frac{1}{x}dx$$

$$= \frac{x^4\ln x}{4} - \frac{1}{4}\int x^3\,dx$$

$$= \frac{x^4\ln x}{4} - \frac{1}{4}\cdot\frac{x^4}{4} + c$$

$$= \frac{x^4}{16}(4\ln x - 1) + c$$

6. $\int \underset{II}{x^4}\cdot \underset{I}{\ln x}\,dx$

$$= \ln x\cdot\frac{x^5}{5} - \int\frac{x^5}{5}\cdot\frac{1}{x}dx$$

$$= \frac{x^5\ln x}{5} - \frac{1}{5}\int x^4\,dx$$

$$= \frac{x^5\ln x}{5} - \frac{1}{5}\cdot\frac{x^5}{5} + c$$

$$= \frac{x^5}{25}(5\ln x - 1) + c$$

7. $\int \underset{I}{x^2}\ \underset{II}{\sin x}\,dx$

$$= x^2\cdot(-\cos x) - \int(-\cos x)\cdot(2x)\,dx$$

$$= -x^2\cos x + 2\int \underset{II}{\cos x}\cdot\underset{I}{x}\,dx$$

$$= -x^2\cos x + 2\left[x\cdot\sin x - \int\sin x\cdot 1\,dx\right]$$

$$= -x^2\cos x + 2x\sin x - 2\int\sin x\,dx$$

$$= -x^2\cos x + 2x\sin x - 2(-\cos x) + c$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

$$= \cos x\left(2 - x^2\right) + 2x\sin x + c$$

8. $\int \underset{II}{x^2}\cdot \underset{I}{\ln 5x}\,dx$

$$= \ln 5x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{5x}(5)\,dx$$

$$= \frac{x^3\ln 5x}{3} - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3\ln 5x}{3} - \frac{1}{3}\cdot\frac{x^3}{3} + c$$

$$= \frac{x^3}{9}(3\ln 5x - 1) + c$$

9. $\int \underset{II}{(x+2)}\ \underset{I}{\ln x}\,dx$

$$= \ln x\cdot\left(\frac{x^2}{2} + 2x\right) - \int\left(\frac{x^2}{2} + 2x\right)\cdot\frac{1}{x}\,dx$$

$$= \frac{\left(x^2 + 4x\right)\ln x}{2} - \frac{1}{2}\left(\frac{x^2}{2} + 4x\right) + c$$

$$= \frac{\left(x^2 + 4x\right)\ln x}{2} - \frac{1}{2}\left(\frac{x^2 + 8x}{2}\right) + c$$

$$=\frac{1}{4}\left[2\left(x^2+4x\right)\ln x-\left(x^2+8x\right)\right]+c$$

$$=\ln x\left(\frac{x^2+4x}{2}\right)-\int\left(\frac{x^2+4x}{2}\right)\cdot\frac{1}{x}\,dx$$

$$=\frac{\left(x^2+4x\right)\ln x}{2}-\frac{1}{2}\int(x+4)\,dx$$

10. $\int \underset{II}{x^n}\ \underset{I}{\ln x}\,dx$

$$=\ln x.\frac{x^{n+1}}{n+1}-\int\frac{x^{n+1}}{n+1}.\frac{1}{x}dx$$

$$=\frac{x^{n+1}}{n+1}\ln x-\frac{1}{n+1}\int x^n\,dx$$

$$=\frac{x^{n+1}}{n+1}\ln x-\frac{1}{n+1}.\frac{x^{n+1}}{n+1}+c$$

$$=\frac{x^{n+1}}{n+1}\ln x-\frac{x^{n+1}}{(n+1)^2}+c$$

11. $\int \underset{I}{x^2}\cdot\underset{II}{e^{-3x}}\,dx$

$$=x^2\cdot\frac{e^{-3x}}{-3}-\int\frac{e^{-3x}}{-3}\cdot(2x)\,dx$$

$$=\frac{x^2e^{-3x}}{-3}+\frac{2}{3}\int \underset{II}{e^{-3x}}\cdot\underset{I}{x}\,dx$$

$$=\frac{x^2e^{-3x}}{-3}+\frac{2}{3}\left[x\cdot\frac{e^{-3x}}{-3}-\int\frac{e^{-3x}}{-3}\cdot 1\,dx\right]$$

$$=-\frac{x^2e^{-3x}}{3}-\frac{2}{9}xe^{-3x}+\frac{2}{9}\int e^{-3x}\,dx$$

$$=-\frac{x^2e^{-3x}}{3}-\frac{2}{9}xe^{-3x}+\frac{2}{9}.\frac{e^{-3x}}{-3}+c$$

$$=-\frac{x^2e^{-3x}}{3}-\frac{2}{9}xe^{-3x}-\frac{2}{27}e^{-3x}+c$$

12. $\int(\ln x)^2\,dx=\int \underset{II}{1}\cdot\underset{I}{(\ln x)^2}\,dx$

$$=(\ln x)^2\cdot x-\int x\cdot 2(\ln x)\cdot\frac{1}{x}\,dx$$

$$=x(\ln x)^2-2\int\ln x\,dx$$

$$=x(\ln x)^2-2\int \underset{II}{1}\cdot\underset{I}{\ln x}\,dx$$

$$=x(\ln x)^2-2\int\ln x\cdot x-\int x\cdot\frac{1}{x}\,dx$$

$$=x(\ln x)^2-2\left[x\ln x-\int dx\right]$$

$$=x(\ln x)^2-2[x\ln x-x]+c$$

$$=-\frac{e^{-3x}}{27}\left(9x^2+6x+2\right)+c$$

13. $\int\frac{\ln 4x}{x^2}\,dx=\int \underset{II}{x^{-2}}\cdot\underset{I}{\ln 4x}\,dx$

$$=\ln 4x\cdot\frac{x^{-1}}{-1}-\int\frac{x^{-1}}{-1}\cdot\frac{1}{4x}\cdot 4\,dx$$

$$=-\frac{\ln 4x}{x}+\int x^{-2}\,dx$$

$$=-\frac{\ln 4x}{x}+\frac{x^{-1}}{-1}+c=-\frac{\ln 4x}{x}-\frac{1}{x}+c$$

$$=-\frac{1}{x}(\ln 4x+1)+c$$

14. $\int e^{-x}\sin 2x\,dx$

Let $I=\int \underset{II}{e^{-x}}\ \underset{I}{\sin 2x}\,dx$

$$=\sin 2x\cdot\frac{e^{-x}}{-1}-\int\frac{e^{-x}}{-1}\cdot\cos 2x(2)\,dx$$

$$=-e^{-x}\sin x+2\int \underset{II}{e^{-x}}\cdot\underset{I}{\cos 2x}\,dx$$

$$=-e^{-x}\sin x+2\left[\cos 2x\cdot\frac{e^{-x}}{-1}-\int\frac{e^{-x}}{-1}\cdot(-\sin 2x)(2)\,dx\right]$$

$$=-e^{-x}\sin 2x-2e^{-x}\cos 2x-4\int e^{-x}\sin 2x\,dx$$

$$I=-e^{-x}\sin 2x-2e^{-x}\cos 2x-4I$$

$$5I=-e^{-x}\sin 2x-2e^{-x}\cos 2x$$

$$=-e^{-x}(\sin 2x+2\cos 2x)$$

$$I=-\frac{e^{-x}}{5}(\sin 2x+2\cos 2x)+c$$

15. $\int e^{2x} \cdot \sin 4x \, dx$

Let $I = \int \underset{II}{e^{2x}} \ \underset{I}{\sin 4x} \, dx$

$= \sin 4x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot \cos 4x (4) \, dx$

$= \frac{e^{2x} \sin 4x}{2} - 2 \int \underset{II}{e^{2x}} \cdot \underset{I}{\cos 4x} \, dx$

$= \frac{e^{2x} \sin 4x}{2} - 2 \left[\cos 4x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot (-\sin 4x)(4) \, dx \right]$

$= \frac{e^{2x} \sin 4x}{2} - e^{2x} \cos 4x - 4 \int e^{2x} \sin 4x \, dx$

$I = \frac{e^{2x}}{2} (\sin 4x - 2 \cos 4x) - 4I$

$5I = \frac{e^{2x}}{2} (\sin 4x - 2 \cos 4x)$

$I = \frac{e^{2x}}{10} (\sin 4x - 2 \cos 4x) + c$

16. $\int e^{2x} \cos 3x \, dx$

Let $I = \int \underset{II}{e^{2x}} \ \underset{I}{\cos 3x} \, dx$

$= \cos 3x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot (-\sin 3x)(3) \, dx$

$= \frac{e^{2x} \cos 3x}{2} + \frac{3}{2} \int \underset{II}{e^{2x}} \cdot \underset{I}{\sin 3x} \, dx$

$= \frac{e^{2x} \cos 3x}{2} + \frac{3}{2} \left[\sin 3x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot (\cos 3x)(3) \, dx \right]$

$= \frac{e^{2x} \cos 3x}{2} + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} \int e^{2x} \cdot (\cos 3x) \, dx$

$I = \frac{e^{2x} \cos 3x}{2} + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} I$

$I + \frac{9}{4} I = \frac{e^{2x} \cos 3x}{2} + \frac{3}{4} e^{2x} \sin 3x$

$\frac{13}{4} I = \frac{e^{2x}}{4} (2 \cos 3x + 3 \sin 3x) + c$

$I = \frac{e^{2x}}{13} (2 \cos 3x + 3 \sin 3x) + c$

17. $\int \sqrt{x} \sin \sqrt{x} \, dx$

Let $x = t^2$, then $dx = 2t dt$

Substituting, we have given integral.

$= \int t \sin t \cdot 2t \, dt$

$= 2 \int \underset{I}{t^2} \ \underset{II}{\sin t} \ dt$

$= 2 \left[t^2 \cdot (-\cos t) - \int (-\cos t)(2t) \, dt \right]$

$= -2t^2 \cos t + 4 \int \underset{II}{\cos t} \cdot \underset{I}{t} \, dt$

$= -2t^2 \cos t + 4 \left[t \cdot \sin t - \int \sin t \cdot 1 \, dt \right]$

$= -2t^2 \cos t + 4t \cdot \sin t - 4 \int \sin t \, dt$

$= -2t^2 \cos t + 4t \sin t - 4(-\cos t) + c$

$= -2t^2 \cos t + 4t \sin t + 4 \cos t + c$

$= 2 \cos t (2 - t^2) + 4t \sin t + c$

$= 2 \cos \sqrt{x} (2 - x) + 4 \sqrt{x} \sin \sqrt{x} + + c$

18. $\int \sqrt{x} . e^{-\sqrt{x}} \, dx$

Let $I = \int \sqrt{x} . e^{-\sqrt{x}} \, dx$

Let $\sqrt{x} = t$, then

$\frac{1}{2\sqrt{x}} dx = dt \Rightarrow dx = 2t \, dt$

$$I = \int t.e^{-t}.2t\,dt$$

$$= 2\int \underset{I}{t^2}\,\underset{II}{e^{-t}}\,dt$$

$$= 2\left[t^2.\frac{e^{-t}}{-1} - \int \frac{e^{-t}}{-1}.2t\,dt\right]$$

$$= -2t^2e^{-t} + 4\int e^{-t}.t\,dt$$

$$= -2t^2e^{-t} + 4\left[t.\frac{e^{-t}}{-1} - \int \frac{e^{-t}}{-1}.1\,dt\right]$$

$$= -2t^2e^{-t} - 4te^{-t} + 4\int e^{-t}\,dt$$

$$= -2t^2e^{-t} - 4te^{-t} - 4e^{-t} + c$$

$$= -2\left(t^2e^{-t} + 2te^{-t} + 2e^{-t}\right) + c$$

$$= -2\left(xe^{-\sqrt{x}} + 2\sqrt{x}e^{-\sqrt{x}} + 2e^{-\sqrt{x}}\right) + c$$

19. $\int x^5e^{x^3}\,dx = \int x^3e^{x^3}.x^2\,dx$

Let $x^3 = t$, then $3x^2\,dx = dt \Rightarrow x^2dx = \frac{1}{3}dt$

$$= \int t.e^t.\frac{1}{3}dt$$

$$= \frac{1}{3}\int \underset{I}{t}\,\underset{II}{e^t}\,dt$$

$$= \frac{1}{3}\left[t.e^t - \int e^t.1\,dt\right] = \frac{1}{3}\left[te^t - e^t\right] + c$$

$$= \frac{1}{3}e^t(t-1) + c = \frac{1}{3}e^{x^3}\left(x^3 - 1\right) + c$$

20. $\int \sec^3 x\,dx$

Let $I = \int \sec^3 x\,dx$

$$= \int \underset{II}{\sec^2 x}.\underset{I}{\sec x}\,dx$$

$$= \sec x.\tan x - \int \tan x.(\sec x\tan x)\,dx$$

$$= \sec x\tan x - \int \sec x\tan^2 x\,dx$$